MATH 1007G **Term test #3 – solution** **3 March 2014**

This test was written in room: ________________ Total marks: **35** marks

This test is closed book. No calculators or electronic aids are permitted. **Please supply your answers on this sheet.**

PLEASE PRINT ________________ ________________ ________________

First name Last name Student number

Please show your work where appropriate! TA's have extra paper if you need it. Test duration: 50 minutes.

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1. Let $f(x) = \begin{cases} \sqrt{4-x}+1, & \text{if } x<4 \\ 9-ax, & \text{if } x \ge 4 \end{cases}$

a. [1] Determine $f(4)$. $f(4) = 9 - a(4) = 9 - 4a$

b. [3] Determine $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} [\sqrt{4-x}+1] \xrightarrow{D.S.} \sqrt{4-4}+1 = \boxed{1}$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} [9-ax] \xrightarrow{D.S.} \boxed{9-4a}$

c. [2.5] For which value of a is f continuous?

if $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$, then f is continuous.

$\therefore \; 9-4a = 1 \Longrightarrow \boxed{a=2}$

2. [2.5] For the case $f(x) = \dfrac{x}{x^2+2}$ determine $D.Q._{[1,4]}$ $\dfrac{f(4)-f(1)}{4-1} = \dfrac{\frac{4}{4^2+2} - \frac{1}{1^2+2}}{3} =$

$\ldots = \dfrac{4/18 - 1/3}{3} = \dfrac{1}{3}\left(\dfrac{4-6}{18}\right) = \boxed{-\dfrac{1}{27}}$

3. [3] Each of the following limits represents the derivative of a particular function, evaluated at a specific point. Determine the function and the point (i.e. $f(x)$ and x_0).

a. $\lim_{h\to 0}\left(\dfrac{4^{3+h}-64}{h}\right)$ $f(x) = 4^x$; $x_0 = 3$

b. $\lim_{z\to \pi/4}\left(\dfrac{\tan z - 1}{z - \pi/4}\right)$ $f(x) = \tan x$; $x_0 = \pi/4$

4. [5] Fill in the blanks

a. $\lim_{x\to 0^-}\left(e^{1/x}\right) =$ $\underline{0}$

b. $\lim_{x\to -\infty}\left(\dfrac{2x^3-3x^2+7}{12-5x-6x^3}\right) =$ $\underline{-1/3}$

c. $\lim_{x\to 0^+}\left(e^{1/x}\right) =$ $\underline{+\infty}$

d. $\lim_{x\to -\infty} \arctan x =$ $\underline{-\pi/2}$

e. $\lim_{x\to \infty}\left(\frac{\cos x}{x^2}\right) =$ $\underline{0}$

5. Determine the following limits if they exist. If a limit does not exist, then determine whether it is $+\infty$, $-\infty$, or neither.

a. [3.5] $\lim_{x\to -\infty} \dfrac{\sqrt{9x^2-x+3}}{x+7} \xrightarrow{D.S.} \dfrac{\infty}{\infty}$

$= \lim_{x\to -\infty} \dfrac{\sqrt{x^2(9-1/x+3/x^2)}}{x(1+7/x)} = \lim_{x\to -\infty} \dfrac{\cancel{(-x)}\cdot\sqrt{9-1/x+3/x^2}}{\cancel{(x)}\cdot(1+7/x)}$

(b.c. for $x<0$, $\sqrt{x^2} = -x > 0$)

$\ldots = \lim_{x\to -\infty} -\dfrac{\sqrt{9-1/x+3/x^2}}{1+7/x} \xrightarrow{D.S.} -\dfrac{\sqrt{9}}{1} = \boxed{-3}$

b. [2.5] $\lim_{x\to -2^+} \dfrac{2x}{x+2} \xrightarrow{D.S.} = \dfrac{A(\neq 0)}{0} \ldots$ so infinite limit

$$\lim_{x\to -2^+} \frac{2x}{x+2} \xrightarrow{D.S.} \frac{2(-2^+)}{-2^+ + 2} = \frac{-4}{0^+} \Rightarrow \frac{(-)}{(+)} \Rightarrow -\infty$$

c. [4] $\lim_{x\to +\infty} 6x - \sqrt{36x^2+12x} \xrightarrow{D.S.} \infty - \infty$

$$\ldots = \lim_{x\to\infty} 6x - \sqrt{36x^2+12x} \cdot \frac{6x+\sqrt{36x^2+12x}}{6x+\sqrt{36x^2+12x}} =$$

$$\ldots = \lim_{x\to\infty} \frac{36x^2-(36x^2+12x)}{6x+\sqrt{36x^2+12x}} = \lim_{x\to\infty} \frac{-12x}{6x+\sqrt{x^2(36+12/x)}} =$$

$$\ldots = \lim_{x\to\infty} \frac{-12x}{6x+x\cdot\sqrt{36+12/x}} = \lim_{x\to\infty} \frac{-12}{6+\sqrt{36+\frac{12}{x}}}$$

$\left[x=\sqrt{x^2} \text{ for } x>0 \right]$

$$\xrightarrow{D.S.} \frac{-12}{6+\sqrt{36}} = \boxed{-1}$$

d. [2.5] $\lim_{x\to +\infty} \left(\dfrac{-4x^3+7x-21}{2-4x-6x^2} \right) \xrightarrow{D.S.} \dfrac{\infty}{\infty}$; but deg (num.) > deg. (denom.)

so $\lim = \pm\infty$ ... which one?

$$\lim_{x\to\infty} \left(\frac{-4x^3+7x-21}{2-4x-6x^2} \right) = \lim_{x\to\infty} \left(\frac{-4x^3}{-6x^2} \right) = \lim_{x\to\infty} \left(\frac{2}{3}x \right) \longrightarrow \boxed{+\infty}$$

$$\therefore \lim_{x\to+\infty} \left(\frac{-4x^3+7x-21}{2-4x-6x^2} \right) = +\infty$$

6. [2.5] Show that the function $f(x) = x^3 + 2x - 1$ has a least one root on [0, 1]. Justify your claim using a theorem covered in class.

1 we note that $f(0) = -1$ and $f(1) = 2$. 2 f is a polynomial and therefore continuous. 3 This means that, by the intermediate value theorem, it takes on all values between -1 and 2 somewhere on $[0,1]$, 4 in particular the value 0. 5 Hence f has a root on $[0,1]$

7. [3] Let $f(x) = \frac{1}{x^2}$. Determine $f'(x)$ using the **limit definition (basic definition)** of the derivative. Simplify the answer.

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0}\left[\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\right] = \lim_{h\to 0}\left[\frac{x^2-(x+h)^2}{hx^2(x+h)^2}\right] =$$

$$\ldots = \lim_{h\to 0}\left[\frac{x^2-x^2-2xh-h^2}{h\,x^2(x+h)^2}\right] = \lim_{h\to 0}\left[\frac{\cancel{h}[-2x-h]}{\cancel{h}\,x^2(x+h)^2}\right] =$$

$$\ldots = \lim_{h\to 0}\left[\frac{-2x-h}{x^2(x+h)^2}\right] \xrightarrow{D.S.} \frac{-2x}{x^2\cdot x^2} = \boxed{\frac{-2}{x^3}}$$