# Solutions to Test 4 - MATH 1107C - Winter 2010

**Version 2**

## PART I: Multiple choice questions (3 points each)

**Choose and circle only one answer. No partial marks here.**
**No justification is required.**

**1.** Let $A$, $B$ and $C$ be $5 \times 5$ matrices such that:

- $B$ is obtained from $A$ by interchanging Row 1 and Row 3;
- $C$ is obtained from $B$ by multiplying all entries in Row 2 of $B$ by 4;
- $\det C = 8$.

What is $\det A$?

**(a)** $-32$ **(b)** $-2$ **(c)** $2$ **(d)** $32$

**Solution:** (b)

**2.** Let $A$, $B$ and $C$ be $3 \times 3$ matrices such that $\det A = 12$, $\det B = 2$ and $\det C = 5$. What is $\det(6A^{-1}B^TC)$?

**(a)** 120 **(b)** 20 **(c)** 720 **(d)** 180

**Solution:** (d)

**3.** Let $S_1 = \{(1,1,3),(4,7,0)\}$, $S_2 = \text{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}\right)$, $S_3 = \text{span}\{(1,3,-1),(2,-1,4)\}$, and $S_4 = \{(r+s,1,r-2) : r \text{ and } s \text{ are real numbers}\}$. The following are subspaces of $\mathbb{R}^3$:

**(a)** $S_1$ and $S_2$ only **(b)** $S_1$ and $S_4$ only **(c)** $S_2$ and $S_3$ only **(d)** $S_3$ and $S_4$ only

**Solution:** (c)

**4.** Let $S_1 = \left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$, $S_2 = \left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \end{bmatrix}\right\}$, $S_3 = \left\{\begin{bmatrix} 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \end{bmatrix}\right\}$, $S_4 = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \end{bmatrix}\right\}$, and $S_5 = \left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$. The following sets are **linearly independent**:

**(a)** $S_1$ and $S_3$ only **(b)** $S_1$ and $S_5$ only **(c)** $S_1$, $S_2$ and $S_5$ only **(d)** $S_3$ and $S_4$ only

**Solution:** This question has been cancelled. The only linearly independent set is $S_5$.

**PART II: Long answer questions**
**Show all your work.**

**5.** Let $A = \begin{bmatrix} -1 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

**(a)** [**5 points**] Find the adjugate of $A$.

**(b)** [**3 points**] Find the determinant of $A$.

**(c)** [**3 points**] Find the inverse of $A$ using the adjugate formula. No other method will be accepted here.

**(d)** [**4 points**] Let $AX = B$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$. Use Cramer's rule to find $x_2$.

**Solution:** **(a)** $\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 3 & 0 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 2 \\ -3 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}^T = \begin{bmatrix} -1 & -3 & 0 \\ -1 & -1 & 0 \\ 2 & 2 & -2 \end{bmatrix}$

**(b)** $\det A = -2$

**(c)** $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = -\frac{1}{2} \begin{bmatrix} -1 & -3 & 0 \\ -1 & -1 & 0 \\ 2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 \\ 1/2 & 1/2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

**(d)** $x_2 = \frac{\det A_2}{\det A} = \frac{\det \begin{bmatrix} -1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{-2} = \frac{-4}{-2} = 2$

**6.** Let $S = \{(x, y, z) \colon 3x + 2y + z = 0\}$ be a subset of $\mathbb{R}^3$.

**(a)** [**5 points**] Show that $S$ is a subspace of $\mathbb{R}^3$.

**(b)** [**5 points**] Find a basis for $S$.

**Solution: (a)**

$$\begin{aligned} S &= \{(x,y,z)\colon 3x+2y+z=0\} \\ &= \left\{(x,y,z)\colon x = -\frac{2}{3}y - \frac{1}{3}z\right\} \\ &= \left\{\left(-\frac{2}{3}y - \frac{1}{3}z, y, z\right) : y,z \in \mathbb{R}\right\} \\ &= \left\{y\left(-\frac{2}{3},1,0\right) + z\left(-\frac{1}{3},0,1\right) : y,z\in\mathbb{R}\right\} \\ &= \operatorname{span}\left\{\left(-\frac{2}{3},1,0\right),\left(-\frac{1}{3},0,1\right)\right\} \\ &\text{or } \operatorname{span}\{(-2,3,0),(-1,0,3)\} \end{aligned}$$

Since any span is a subspace, $S$ is a subspace.

**(b)** $\left\{\left(-\frac{2}{3},1,0\right),\left(-\frac{1}{3},0,1\right)\right\}$ is a basis of $S$, since it is a spanning set for $S$ (by (a)) and it is linearly independent.

$\{(-2,3,0),(-1,0,3)\}$ is another possible basis of $S$.

**7.** [**8 points**] Verify whether or not the following set of vectors is a basis of $\mathbb{R}^3$:

$$\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}-1\\0\\2\end{bmatrix}\right\}.$$

Explain clearly.

**Solution:** $\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&2\end{bmatrix} \xrightarrow{R_2\leftarrow R_2+R_1} \begin{bmatrix}1&0&-1\\0&1&-1\\0&-1&2\end{bmatrix} \xrightarrow{R_3\leftarrow R_3+R_2} \begin{bmatrix}1&0&-1\\0&1&-1\\0&0&1\end{bmatrix}$

This row reduction shows that the set of vectors is linearly independent. We can now conclude that it is a basis of $\mathbb{R}^3$ using one of the following arguments:

*Justification 1*: Since the dimension of $\mathbb{R}^3$ is three, any three linearly independent vectors in $\mathbb{R}^3$ form a basis of $\mathbb{R}^3$. Therefore the above set is a basis of $\mathbb{R}^3$.

*Justification 2*: By the inverse matrix theorem (or Theorem 3, page 233), the above set spans $\mathbb{R}^3$. Hence it is a basis of $\mathbb{R}^3$.