COMP 232 Mathematics for Computer Science
Fall 2013
Midterm Exam

Name: ______________________ Total Points:

ID: ______________________ ________ / 62

Instructions. This is a closed book exam. The only allowed tool is an ENCS approved calculator. Provide all answers in this booklet. Use pen, not pencil. Do not detach any pages from this exam!

($2^{\text{pts}}_{\text{ea.}}$) 1. Let the universe of discourse be $\mathbb{Z}^+$, the set of positive integers. For each of the following sentences, indicate whether it is true or false. You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0. [ ] 10 pts

(a) $\forall x\big((x < 0) \lor (x \leq 2x)\big)$

☑ True ☐ False ☐ Don't know!

(b) $\exists x \exists y\big((x + y = 0) \lor (x \cdot y = 0)\big)$

☐ True ☑ False ☐ Don't know!

(c) $\forall x \forall y(x \cdot y \geq x + y)$

☐ True ☑ False ☐ Don't know!

(d) $\exists x \exists y\big((x = 3) \lor (y = 4)\big)$

☑ True ☐ False ☐ Don't know!

(e) $\exists x \forall y \exists z\big((y = x + z) \land (z \leq x)\big)$

☐ True ☑ False ☐ Don't know!

[ ] 10 pts

(6 pts ea.) 2. Here you are to prove propositional equivalences using the laws in the handout.

12 pts

(a) Here is a proof that $p \to (q \to r) \equiv (p \wedge q)$.

Step			Law applied
$p \to (q \to r)$	$\equiv$	$\neg p \vee (q \to r)$	Implication
	$\equiv$	$\neg p \vee (\neg q \vee r)$	Implication
	$\equiv$	$(\neg p \vee \neg q) \vee r$	Associativity
	$\equiv$	$\neg(p \wedge q) \vee r$	de Morgan
	$\equiv$	$(p \wedge q) \to r$	Implication

In the rightmost column above, fill in the law applied for each step (see handout for a list of laws)

(b) In the table below, construct a proof of the equivalence

$$(r \vee p) \to (r \vee q) \equiv r \vee (p \to q)$$

similarly to (a).

Step			Law applied
$(r \vee p) \to (r \vee q)$	$\equiv$	$\neg(r \vee p) \vee (r \vee q)$	Implication
	$\equiv$	$\Big((\neg r) \wedge (\neg p)\Big) \vee (r \vee q)$	de Morgan
	$\equiv$	$\Big(\Big((\neg r) \wedge (\neg p)\Big) \vee r\Big) \vee q$	Associativity
	$\equiv$	$\Big(r \vee \Big((\neg r) \wedge (\neg p)\Big)\Big) \vee q$	Commutativity
	$\equiv$	$\Big(\Big(r \vee (\neg r)\Big) \wedge \Big(r \vee (\neg p)\Big)\Big) \vee q$	Distributivity
	$\equiv$	$\Big(\Big(r \vee (\neg p)\Big) \wedge \Big(r \vee (\neg r)\Big)\Big) \vee q$	Commutativity
	$\equiv$	$\Big(\Big(r \vee (\neg p)\Big) \wedge T\Big) \vee q$	Excluded middle
	$\equiv$	$\Big(r \vee (\neg p)\Big) \vee q$	Identity
	$\equiv$	$r \vee \Big((\neg p) \vee q\Big)$	Associativity
	$\equiv$	$r \vee (p \to q)$	Implication

12 pts

(4pts) 3. The negation of the statement $\forall x \neg \forall y \exists z (P(x,z) \wedge Q(z,y))$ is

4 pts

- [x] $\exists x \forall y \exists z (P(x,z) \wedge Q(z,y))$
- [ ] $\forall x \exists y \forall z (\neg P(x,z) \vee \neg Q(z,y))$
- [ ] $\forall x \forall y \exists z (\neg P(x,z) \wedge \neg Q(z,y))$
- [ ] $\forall x \exists y \forall z (\neg P(x,z) \wedge \neg Q(z,y))$
- [ ] $\exists x \exists y \forall z (P(x,z) \vee \neg Q(z,y))$

(4pts) 4. Which of the following statements is the contrapositive of the statement "You win the game if you know the rules but are not overconfident."

4 pts

- [ ] "If you don't know the rules or are overconfident then you lose the game."
- [ ] "A necessary condition that you know the rules or you are not overconfident is that you win the game."
- [ ] "If you don't know the rules and are overconfident then you win the game."
- [x] "If you lose the game then you don't know the rules or you are overconfident."
- [ ] "A sufficient condition that you win the game is that you know the rules or you are not overconfident."

(4pts) 5. To prove $p \wedge (\neg q) \Rightarrow r \vee (\neg s)$ by contradiction, which of the following propositions is the appropriate one to prove.

4 pts

- [x] $\big(p \wedge (\neg q) \wedge (\neg r) \wedge s\big) \Rightarrow False$
- [x] $\big((\neg q) \wedge p \wedge (\neg r) \wedge s\big) \Rightarrow False$
- [x] $\big((\neg q) \wedge p \wedge s \wedge (\neg r)\big) \Rightarrow False$
- [ ] $\big((\neg p) \wedge q \wedge r \wedge (\neg s)\big) \Rightarrow False$
- [ ] $\big((\neg p) \wedge q \wedge s \wedge (\neg r)\big) \Rightarrow False$

(4pts) 6. Consider the assertion

4 pts

$$\exists x \in \mathbb{Z}\Big(P(x) \wedge Q(x)\Big) \equiv \Big(\exists x \in \mathbb{Z}(P(x))\Big) \wedge \Big(\exists x \in \mathbb{Z}(Q(x))\Big)$$

Which of the following statements correctly describes the assertion?

- [ ] The assertion is true. The proof follows from the distributive laws for $\wedge$
- [ ] The assertion is false. Counterexample: $P(x)$ means "x is divisible by 6," and $Q(x)$ means "x is divisible by 3."
- [x] The assertion is false. Counterexample: $P(x)$ means "" $x < 0$," and $Q(x)$ means "$x \geq 0$."
- [ ] The assertion is true. To see why, let $P(x)$ mean "x is divisible by 6," and $Q(x)$ mean "x is divisible by 3." If $x = 6$, then x is divisible by both 3 and 6, so both side of the equivalence have the same truth value for this x.
- [ ] The assertion is false. Counterexample: $P(x)$ means '"$x < 0$ is a square" and $Q(x)$ means "x is odd."

16 pts

(6 pts ea.) 7. (a) Consider the assertion "Let a and b be positive integers. If $a < b$ then $a < \frac{a+b}{2} < b$." Give a direct proof of the assertion.

24 pts

Solution:

$a \leq 2a = a + a < a + b \;\Rightarrow\; a < \frac{a+b}{2}$

$\frac{a+b}{2} < \frac{b+b}{2} = \frac{2b}{2} = b \;\Rightarrow\; \frac{a+b}{2} < b$

(b) Consider the assertion "Let $n \in \mathbb{Z}$. If $n^5 + 7$ is even, then n is odd." Give a indirect proof of the assertion.

Solution: We need to prove $even(n) \Rightarrow odd(n^5 + 7)$

$even(n) \Rightarrow n = 2k$, for some $k \in \mathbb{Z}$

$\Rightarrow n^5 + 7 = (2k)^5 + 7 = 32k^5 + 7 = 32k^5 + 6 + 1 = 2(16k^5 + 3) + 1 \;\Rightarrow odd(n^5 + 7)$

(c) Consider the assertion "Let $x, y \in \mathbb{R}$ with $x \geq 0$ and $y \geq 0$. Then $\frac{x+y}{2} \geq \sqrt{xy}$." Prove the assertion by contradiction.

Solution: Suppose to the contrary that $\frac{x+y}{2} < \sqrt{xy}$.

Then $\frac{x+y}{2} \cdot \frac{x+y}{2} < \sqrt{xy} \cdot \sqrt{xy} = xy$

$\Rightarrow (x + y)^2 < 4xy$

$\Rightarrow x^2 + 2xy + y^2 < 4xy$

$\Rightarrow x^2 - 2xy + y^2 < 0$

$\Rightarrow (x - y)^2 < 0$; a contradiction, since a square always is ≥ 0.

(d) Consider the assertion "For all intgers n it holds that $n^2 + n$ is even."
Give a proof by cases of the assertion.

Solution:

- Case 1: n is even.

$\Rightarrow n = 2k$, for some $k \in \mathbb{Z} \Rightarrow n^2 + n = 4k^2 + 2k = 2(2k^2 + k) \ \Rightarrow n^2 + n$ is even

- Case 2: n is odd.

$\Rightarrow n = 2k + 1$, for some $k \in \mathbb{Z}$

$\Rightarrow n^2 + n = (2k+1)^2 + (2k+1) = (4k^2 + 4k + 1) + (2k + 1) = 4k^2 + 6k + 2$

$\Rightarrow n^2 + n = 2(2k^2 + 3k + 1)$

$\Rightarrow n^2 + n$ is even

—... End of Exam ...—

0 pts