University of Ottawa

Department of Mathematics and Statistics

Final Exam - MAT1302 E - Fall 2014

Professor: Aziz Khanchi

Surname ____________________ First Name ____________________

Student # ____________________ Seat # ____________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) This exam consists of three parts. Questions 1 through 3 are **answer-only**. For these questions, only your final answer will be considered for marks. Questions 4 through 14 are multiple choice questions. For these questions write your answer in the table on the second page. Questions 15 through 22 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

(c) The number of points available for each question is indicated in square brackets.

(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(f) You may use the last page of the exam as scrap paper.

Good luck!

Please answer multiple choice questions in this table.

Question	Answer	Question	Answer
1	**X**	11	
2	**X**	12	
3	**X**	13	
4		14	
5			
6			
7			
8			
9			
10			

Please do not write in the table below.

Question	1–3	4–14	15	16	17	18	19	20	21	22	Total
Maximum	4.5	16.5	3	5	4	6	4	4	4	4	55
Grade											

Part A: Answer-Only Questions (1.5 points each)

Question 1. Suppose that A, B, C and X are $n \times n$ matrices, where A, B and C are invertible. Solve the following equation for the matrix X, i.e., express X in terms of A, B, C, their inverses and their transposes.

$$CB^{-1}(X + A)B^T = A$$

Answer:
Solution: $X = BC^{-1}A(B^T)^{-1} - A$.

Question 2. What are the eigenvalues of the matrix $\begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$?

Answer:
Solution: $\pm 2i$

Question 3. Suppose that $z = -3 + 4i$. If the number $z\bar{z}$ is written in the form $a + bi$, with $a, b \in \mathbb{R}$, what are a and b?

Answer: $a=$ $b=$
Solution: $a = 25, b = 0$

Part B: Multiple Choice Questions. Write your answers in the table on the second page. 1.5 points each.

Question 4. Let $A = \begin{bmatrix} -7 & -2 & 27 & 8 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \sqrt{71} \\ 0 & 0 & -4 & 0 & 25 \\ 0 & 0 & 0 & 15 & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{14} \end{bmatrix}$. What is det A?

(a) 35 (b) 5 (c) 20 (d) 0 (e) 10

Solution: b

Question 5. Let $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Which one is a basis for the null space of B?

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 0 \\ 1 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Solution: (d)

Question 6. Let $A = \begin{bmatrix} 1 & -2 \\ -2 & 3 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 2 \\ -1 & 0 \end{bmatrix}$. Which one is the matrix $(A^T(2B))^2$?

(a) $B = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (c) $B = \begin{bmatrix} 16 & 4 \\ -4 & 16 \end{bmatrix}$

(d) $B = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$ (e) $B = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

Solution: (d)

Question 7. Let A be $n \times n$ invertible matrix. Which of the following statements are true?

(i) The matrix A is row equivalent to the identity matrix.
(ii) The equation $A\vec{x} = \vec{0}$ has a nontrivial solution.
(iii) The determinant of A is not zero.

(a) All (b) (ii) only (c) (iii) only (d) (i) and (iii) only (e) (i) and (ii) only

Solution: (d)

Question 8. Let A be 5×6 matrix. If the dimension of the column space of A is 2, then the dimension of the null space of A is:

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Solution: (d)

Question 9. Given three subsets of $\mathbb{R}^2$:

$$U = \left\{ \begin{bmatrix} ab \\ a+b \end{bmatrix} |a, b \in \mathbb{R} \right\}, \quad V = \left\{ \begin{bmatrix} a+4b \\ 2a-b \end{bmatrix} |a, b \in \mathbb{R} \right\}, \quad W = \left\{ \begin{bmatrix} b+1 \\ b \end{bmatrix} |b \in \mathbb{R} \right\}.$$

Which of them are **not** subspaces of $\mathbb{R}^2$?

(a) U and V (b) U and W (c) V and W (d) U only (e) V only.

Solution: (b)

Question 10. Suppose A, B and C are 2×2 matrices such that $\det A = 3$, $\det B = 4$ and $\det C = 5$. What is $\det(2A^T B^{-1} C)$?

(a) 1 (b) 24 (c) 120 (d) 30 (e) 15

Solution: (e)

Question 11. Find the values of k such that the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} k \\ -4 \\ -6 \end{bmatrix} \right\}$ is linearly independent.

(a) $k = 2$ (b) $k \neq 2$ (c) $k = -2$ (d) $k \neq -2$ (e) $k \neq 0$

Solution: (d)

Question 12. Which set is a basis for the line in $\mathbb{R}^2$ that passes through origin and the point $(-2, 1)$?

(a) $\left\{ \begin{bmatrix} 10 \\ -5 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 10 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$
(e) $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Solution: (a)

Question 13. Suppose that M is a 10×15 matrix of rank 10. What is the dimension of Nul$A$?

(a) 0 (b) 1 (c) 5 (d) 10 (e) 15
Solution: (a)

Question 14. Suppose that $z = 4 - 2i$ and $w = 2i$. $\dfrac{z}{w}$ is:

(a) $4 - 2i$ (b) $1 + 2i$ (c) $-1 - 2i$ (d) $-1 + 2i$ (e) $1 - 2i$
Solution: (c)

Part C: Long Answer Questions, you have to show your work!

Question 15. [**3 points**] A closed economy consists of two sectors: Auto and Service. To produce one unit, the Auto sector must consume 0.2 units from the Auto sector and 0.4 units from the Service sector. On the other hand, to produce one unit, the Service sector must consume 0.3 units from the Auto sector and 0.6 units from the Service sector.

(a) What is the consumption matrix C for this economy?

(b) Find the production levels that will satisfy the final demand of 10000 units from Auto sector and 2000 units from Service sector.

Solution: (a) The consumption matrix of this economy is: $C = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$.

(b) $I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.3 \\ -0.4 & 0.4 \end{bmatrix}$.

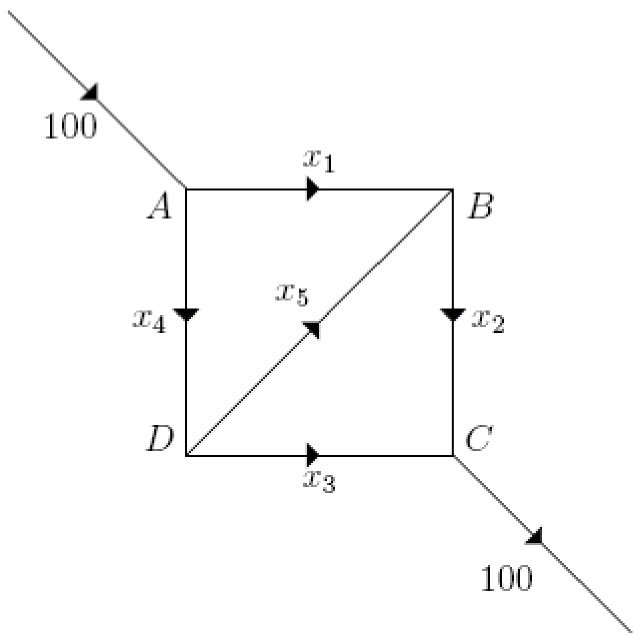
$(I - C)^{-1} = \frac{1}{0.32 - 0.12} \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.8 \end{bmatrix} = \begin{bmatrix} 2 & 1.5 \\ 2 & 4 \end{bmatrix}$.

Note that $\mathbf{d} = \begin{bmatrix} 10000 \\ 2000 \end{bmatrix}$.

$$(I - C)^{-1}\mathbf{d} = \begin{bmatrix} 2 & 1.5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 10000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 23000 \\ 28000 \end{bmatrix}.$$

So, the production levels are: Auto 23000, Service 28000.

Question 16. [**5 points**] The traffic flow in a city is represented by the diagram below. The arrows indicate the direction of one-way traffic.



(a) [**2 points**] Write down a linear system describing this traffic flow. Do not perform any calculations at this stage.

Solution:

Intersection	**Flow in**		**Flow out**
A	100	$=$	$x_1 + x_4$
B	$x_1 + x_5$	$=$	x_2
C	$x_2 + x_3$	$=$	100
D	x_4	$=$	$x_3 + x_5$
Overall	100	$=$	100

(b) [**1 point**] The reduced echelon form of the augmented matrix in part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 100 \\ 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Give the general solution of the system (ignore the constraints on the variables for now).

Solution: Since the fourth and the fifth columns do not contain pivot elements, x_4 and x_5 are free variables. The general solution is:

$$\begin{aligned} x_1 &= 100 - x_4 \\ x_2 &= 100 - x_4 + x_5 \\ x_3 &= x_4 - x_5 \\ x_4, x_5 &\quad \text{free} \end{aligned}$$

(c) **[2 points]** If road DB is closed due to construction and if the flow on road AD is $x_4 = 10$, what is the traffic flow on road BC?

Solution: The two conditions give $x_4 = 10$ and $x_5 = 0$. Substituting these values into the general solution of the system, we obtain $x_2 = 100 - 10 = 90$.

Question 17. [**4 points**] Compute the determinant of $A = \begin{bmatrix} -2 & 3 & 4 & -1 & 5 \\ -1 & 1 & -2 & 3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & -4 & 1 \\ 0 & 1 & 0 & 17 & 0 \end{bmatrix}$

Solution: We expand along the third row:

$$detA = (-1) \begin{vmatrix} -2 & 3 & 4 & 5 \\ -1 & 1 & -2 & -3 \\ -3 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

Now we expand along the forth row:

$$detA = - \begin{vmatrix} -2 & 4 & 5 \\ -1 & -2 & -3 \\ -3 & 2 & 1 \end{vmatrix}$$

Now we expand along the first row:

$$\begin{aligned} detA &= -\left(-2 \begin{vmatrix} -2 & -3 \\ 2 & 1 \end{vmatrix} - 4 \begin{vmatrix} -1 & -3 \\ -3 & 1 \end{vmatrix} + 5 \begin{vmatrix} -1 & -2 \\ -3 & 2 \end{vmatrix} \right) \\ &= -(-2(-2+6) - 4(-1-9) + 5(-2-6)) \\ &= -(-8+40-40) = 8 \end{aligned}$$

Question 18. **[6 points]** Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(a) Find the eigenvalues of A.

Solution:

$$det(A - \lambda I) = (1-\lambda)(-1-\lambda)(-1-\lambda) = (1-\lambda)(-1-\lambda)^2 = (1-\lambda)(1+\lambda)^2.$$

Hence, the eigenvalues of A are 1 and -1.

(b) For each eigenvalue, find a basis for the corresponding eigenspace.

Solution: For $\lambda = 1$ we row reduce:

$$[A - I|0] = \left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow{row\ reduce} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The eigenspace consists of all the vectors of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_1 \in \mathbb{R}$$

Hence, a basis for this eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

For $\lambda = -1$, we row reduce

$$[A - I|0] = \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{row\ reduce} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The eigenspace consists of all the vectors of the form:

$$\vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

Hence, a basis for this eigenspace is given by:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

(You do not need to compute P^{-1}.)

Solution: $P = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Question 19. [**4 points**] Is the matrix $A = \begin{bmatrix} 3 & 12 & -5 \\ 1 & 4 & -2 \\ -2 & -7 & 6 \end{bmatrix}$ invertible? If so, find A^{-1}.

Solution:

$$\left[\begin{array}{ccc|ccc} 3 & 12 & -5 & 1 & 0 & 0 \\ 1 & 4 & -2 & 0 & 1 & 0 \\ -2 & -7 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 4 & -2 & 0 & 1 & 0 \\ 3 & 12 & -5 & 1 & 0 & 0 \\ -2 & -7 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_1 + R_3]{-3R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 4 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 4 & -2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_3 + R_2]{2R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 2 & -5 & 0 \\ 0 & 1 & 0 & -2 & 8 & 1 \\ 0 & 0 & 1 & 1 & -3 & 0 \end{array}\right] \xrightarrow{-4R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -37 & -4 \\ 0 & 1 & 0 & -2 & 8 & 1 \\ 0 & 0 & 1 & 1 & -3 & 0 \end{array}\right]$$

Therefore

$$A^{-1} = \begin{bmatrix} 10 & -37 & -4 \\ -2 & 8 & 1 \\ 1 & -3 & 0 \end{bmatrix}$$

Question 20. [**4 points**] Determine if the following vectors are linearly independent.

$$\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{v_3} = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 0 \end{bmatrix}.$$

Solution: We reduce the matrix whose columns are the vectors.

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the third column is not a pivot, the corresponding system has non-trivial solution. Thus the vectors are linearly dependent.

Question 21. [**4 points**] Let $A = \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 2 & -4 & 4 & -2 \\ 1 & -2 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for Col A.
(b) What is the dimension of the Col$A$?
Solution:

$$\begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 2 & -4 & 4 & -2 \\ -1 & 2 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and third columns are pivot columns. It follows that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\}$ is a basis for Col A. dim col$A = 2$.

Question 22. [**4 points**] Is the following linear system consistent or inconsistent? If it is inconsistent, explain why; if it is consistent, find the general solution.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 2x_4 + 6x_5 \\ -x_1 - x_3 + x_5 + 1 &= 2x_2 - 3x_4 \\ x_1 &= -2x_2 - 2x_3 + 10x_5 \end{aligned}$$

Solution: We first write the system in standard form:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & - & 2x_4 & - & 6x_5 & = & 0 \\ -x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & + & x_5 & = & -1 \\ x_1 & + & 2x_2 & + & 2x_3 & & & - & 10x_5 & = & 0 \end{array}$$

Then write down the augmented matrix and row reduce to RREF:

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 2 & -2 & -6 & 0 \\ -1 & -2 & -1 & 3 & 1 & -1 \\ 1 & 2 & 2 & 0 & -10 & 0 \end{array}\right] \xrightarrow[R_3-R_1\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{rrrrr|r} 1 & 2 & 2 & -2 & -6 & 0 \\ 0 & 0 & 1 & 1 & -5 & -1 \\ 0 & 0 & 0 & 2 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3\to R_3} \left[\begin{array}{rrrrr|r} 1 & 2 & 2 & -2 & -6 & 0 \\ 0 & 0 & 1 & 1 & -5 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_1+2R_3\to R_1]{R_2-R_3\to R_2} \left[\begin{array}{rrrrr|r} 1 & 2 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2\to R_1} \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

Since the last column is not a pivot column, the system is consistent. Solving for the basic variables in terms of the free variables, we obtain the general solution:

$$\begin{aligned} x_1 &= 2 - 2x_2 + 4x_5 \\ x_3 &= -1 + 3x_5 \\ x_4 &= 2x_5 \\ x_2, x_5 & \quad \text{free} \end{aligned}$$

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