MAT 2377 3X (Spring 2011)

Assignment 2 - Solutions

[7] 1.

(a) Since $f(x) \geq 0$ for $x = 0, 1, 2, 3$ and

$$\sum_{x=0}^{3} f(x) = (8/15)(1/2)^0 + (8/15)(1/2)^1 + (8/15)(1/2)^2 + (8/15)(1/2)^3 = 1,$$

then f is a probability mass function.

(b) $P(X \leq 1) = f(0) + f(1) = 4/5 = 0.8$

(c) $P(2 \leq X < 4) = f(2) + f(3) = 1/5 = 0.2$

(d)

$$\begin{aligned} P(X \leq 1 \quad \text{or} \quad X > 2) &= P(X \leq 1) + P(X > 2) \\ &= \frac{4}{5} + \frac{1}{15} = \frac{13}{15} = 0.8667. \end{aligned}$$

(e) The mean of X is

$$\mu_X = \sum_{x=0}^{3} x\, f(x) = 11/15 = 0.73333.$$

The variance of X is

$$\begin{aligned} \sigma_X^2 &= E[X^2] - \mu_X^2 = \left(\sum_{x=0}^{3} x^2\, f(x) \right) - \mu_X^2 \\ &= (7/5) - (11/15)^2 = 0.86222. \end{aligned}$$

The standard deviation is $\sigma_X = \sqrt{\sigma_X^2} = \sqrt{0.86222} = 0.92856.$

(f) The cumulative distribution function (c.d.f) for X is

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= \begin{cases} 0, & x < 0 \\ 8/15, & 0 \leq x < 1 \\ 8/15 + (8/15)(1/2) = 4/5 = 0.8, & 1 \leq x < 2 \\ 4/5 + (8/15)\,(1/2)^2 = 14/15 = 0.93333, & 2 \leq x < 3 \\ 14/15 + (8/15)\,(1/2)^3 = 1, & x \geq 3 \end{cases} \end{aligned}$$

[4] 2.

(a) Since $X \sim B(15, 0.01)$, then

$$\mu_X = n\,p = (15)(0.01) = 0.15 \quad \text{and} \quad \sigma_X = \sqrt{n\,p\,(1-p)} = 0.3854.$$

So the probability that X exceeds its mean by 3 standard deviations is

$$\begin{aligned} & P(X > \mu_X + 3\,\sigma_X) \\ = \;& P(X > 0.15 + 3\,(0.3854)) \\ = \;& P(X > 1.3062) \\ = \;& 1 - P(X \leq 1) \\ = \;& 1 - \left[\binom{15}{0}(0.01)^0(0.99)^{15} + \binom{15}{1}(0.01)^1(0.99)^{14}\right] \\ = \;& 0.00963. \end{aligned}$$

(b) Let Y be the number of samples of size 15 that we require to observe a sample for which X exceeds its mean by more than 3 standard deviations. Y has a geometric distribution with $p = 0.00963$. We want

$$E[Y] = \frac{1}{p} = 103.8422.$$

[2] 3. Let X be the number of defective cards among the sample of size $n = 10$. We want

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{5}{0}\binom{45}{10}}{\binom{50}{10}} = 0.6894.$$

[4] 4.

(a) Let X be the number of correctly transmitted bits among $n = 5$ bits. We have $X \sim B(5, 0.8)$. The expected number of correctly transmitted bits is

$$E[X] = n\,p = 5\,(0.8) = 4.$$

(b) The message will be properly decoded if and only if there are at least 3 properly transmitted bits among the five bits. Thus, the probability that the message is decoded properly is

$$\begin{aligned} & P(X \geq 3) \\ = \;& \binom{5}{3}(0.8)^3(0.2)^2 + \binom{5}{4}(0.8)^4(0.2)^1 + \binom{5}{5}(0.8)^5(0.2)^0 \\ = \;& 0.9421. \end{aligned}$$

[4] 5.

(a) Let X be the number of surface flaws in an automobile's interior. X has a Poisson distribution with mean $\mu = \alpha\, t = (0.045)(10) = 0.45$. The probability that there are no surface flaws in an automobile's interior is

$$P(X = 0) = e^{-0.45}\,\frac{(0.45)^0}{0!} = 0.63763.$$

(b) Let Y be the number of cars with surface flaws among $n = 10$ cars. Y has a binomial distribution with $n = 10$ and $p = 1 - 0.63763 = 0.36237$. Recall that in part (a) we computed the probability that a car will not have any surface flaws.

The probability that at most one of the 10 cars has any surface flaws is

$$\begin{aligned} & P(Y \leq 1) \\ = \; & \binom{10}{0}(0.36237)^0(0.63763)^{10} + \binom{10}{1}(0.36237)^1(0.63763)^9 \\ = \; & 0.07424. \end{aligned}$$

[4] 6.

(a) Let X be the number of cracked eggs among $n = 12$ eggs. X has a binomial distribution with $n = 12$ and $p = 0.01$.

(b) The probability that a carton of a dozen eggs results in more than one cracked egg is

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - \left[\binom{12}{0}(0.01)^0(0.99)^{12} + \binom{12}{1}(0.01)^1(0.99)^{11}\right] \\ &= 0.00617. \end{aligned}$$

(c) Let X be the number of cracked eggs among $n = 1200$ eggs. X has a binomial distribution with $n = 1200$ and $p = 0.01$. Since n is large (i.e. $n \geq 20$) and p is small (i.e. $p \leq 0.05$), then the Poisson approximation to the binomial distribution is good. We want

$$P(X \leq 1) \approx e^{-n\,p}\frac{(n\,p)^0}{0!} + e^{-n\,p}\frac{(n\,p)^1}{1!} = 0.0000799.$$

(d) Let T be the number of eggs that we need to load to observe a cracked egg. T has a geometric distribution with $p = 0.01$. The probability that we need more than 100 loaded eggs to observe a cracked egg is

$$P(T > 100) = 1 - F_T(100) = 1 - \big[1 - (1 - p^{100})\big] = (0.99)^{100} = 0.3660.$$

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