Solutions to Test 4 - MATH 1107C - Winter 2010

Version 1

PART I: Multiple choice questions (3 points each)
Choose and circle only one answer. No partial marks here.
No justification is required.

1. Let A, B and C be 5×5 matrices such that:

- B is obtained from A by interchanging Row 2 and Row 3;
- C is obtained from B by multiplying all entries in Row 1 of B by 2;
- $\det C = 4$.

What is $\det A$?

(a) -2 **(b)** 2 **(c)** -8 **(d)** 8

Solution: (a)

2. Let A, B and C be 3×3 matrices such that $\det A = 2$, $\det B = 3$ and $\det C = 10$. What is $\det(5A^T BC^{-1})$?

(a) 3 **(b)** 60 **(c)** 75 **(d)** 300

Solution: (c)

3. Let $S_1 = \text{span}\{(1, 0, -1), (2, -1, 1)\}$, $S_2 = \text{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right)$, $S_3 = \{(1, 2, 0), (-1, 1, 0)\}$, and $S_4 = \{(a + b, a - b, 1) \colon a \text{ and } b \text{ are real numbers}\}$. The following are subspaces of $\mathbb{R}^3$:

(a) S_1 and S_2 only **(b)** S_1 and S_4 only **(c)** S_2 and S_3 only **(d)** S_3 and S_4 only

Solution: (a)

4. Let $S_1 = \left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \end{bmatrix}\right\}$, $S_2 = \left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix}\right\}$, $S_3 = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$, $S_4 = \left\{\begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$, and $S_5 = \left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$. The following sets are **linearly independent**:

(a) S_1, S_3 and S_5 only **(b)** S_1, S_4 and S_5 only **(c)** S_2 and S_5 only **(d)** S_3 and S_5 only

Solution: (d)

PART II: Long answer questions
Show all your work.

5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

(a) [**5 points**] Find the adjugate of A.

(b) [**3 points**] Find the determinant of A.

(c) [**3 points**] Find the inverse of A using the adjugate formula. No other method will be accepted here.

(d) [**4 points**] Let $AX = B$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Use Cramer's rule to find x_3.

Solution: **(a)** $\text{adj } A = \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -1 & 3 & -1 \\ 1 & 3 & -1 \\ 0 & -2 & 0 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 0 \\ 3 & 3 & -2 \\ -1 & -1 & 0 \end{bmatrix}$

(b) $\det A = -2$

(c) $A^{-1} = \dfrac{1}{\det A}\text{adj } A = -\dfrac{1}{2}\begin{bmatrix} -1 & 1 & 0 \\ 3 & 3 & -2 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & -3/2 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

(d) $x_3 = \dfrac{\det A_3}{\det A} = \dfrac{\det \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}}{-2} = \dfrac{-3}{-2} = \dfrac{3}{2}$

6. Let $S = \{(x, y, z)\colon x + 2y + 3z = 0\}$ be a subset of $\mathbb{R}^3$.

(a) [**5 points**] Show that S is a subspace of $\mathbb{R}^3$.

(b) [**5 points**] Find a basis for S.

Solution: (a)

$$\begin{aligned} S &= \{(x,y,z)\colon x+2y+3z=0\} \\ &= \{(x,y,z)\colon x=-2y-3z\} \\ &= \{(-2y-3z,y,z)\colon y,z\in\mathbb{R}\} \\ &= \{y(-2,1,0)+z(-3,0,1)\colon y,z\in\mathbb{R}\} \\ &= \operatorname{span}\{(-2,1,0),(-3,0,1)\} \end{aligned}$$

Since any span is a subspace, S is a subspace.

(b) $\{(-2,1,0),(-3,0,1)\}$ is a basis of S, since it is a spanning set for S (by (a)) and it is linearly independent.

7. [**8 points**] Verify whether or not the following set of vectors is a basis of $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Explain clearly.

Solution: $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

This row reduction shows that the set of vectors is linearly independent. We can now conclude that it is a basis of $\mathbb{R}^3$ using one of the following arguments:

Justification 1: Since the dimension of $\mathbb{R}^3$ is three, any three linearly independent vectors in $\mathbb{R}^3$ form a basis of $\mathbb{R}^3$. Therefore the above set is a basis of $\mathbb{R}^3$.

Justification 2: By the inverse matrix theorem (or Theorem 3, page 233), the above set spans $\mathbb{R}^3$. Hence it is a basis of $\mathbb{R}^3$.