Ryerson University

Department of Mathematics

Test # 2

MTH 141 – Linear Algebra

Student Number: ________________. Last Name:________________. First Name (**Print**):________________.

Signature:____________________.

Professor (circle one)

C. Grandison B. Tasić

Date: March 12, 2010, 4:00 pm

Duration: 1hr. 30 min.

Solns

Section : _______

Instructions:

1. Have your student card available on your desk.
2. This is a closed-book test. **Notes, calculators and other aids are not permitted.** Verify that your test has pages 1-7.
3. Do not separate the pages of this test booklet.
4. The point value of each question is indicated by the question number.
5. Include all significant steps in your solutions to the questions, presented in the correct order. **Unjustified answers will be given little or no credit.** Cross out or erase all rough work not relevant to your solution.
6. Present your solutions neatly and legibly in the space provided. **Messy or illegible solutions will receive no credit.**
7. If you need more space, use the back of the previous page. Indicate this fact on the original page.

For Instructor's use only.

Question	Mark
1	
2	
3	
4	
5	
6	
Total	/50

1. [(4+3) marks]Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find elementary matrices E, F and G such that $EFGA = I_3$.

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad GA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \qquad F(GA) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E\big(F(GA)\big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I_3$$

(b) Express A as a product of elementary matrices.

By (a), $EFGA = I_3$ so,

$$A = G^{-1}F^{-1}E^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. [(3 + 3) marks] Let A and B be 5×5 matrices with $\det A = -1$ and $\det B = 2$. Compute the following:

(Make sure to show your work and quote any results that you use)

(a) $\det(-(A^2B^T)^{-1})$

$$= (-1)^5 \det\left((A^2B^T)^{-1}\right)$$

$$= -\frac{1}{\det(A^2B^T)}$$

$$= -\frac{1}{\det(A^2)\det(B^T)}$$

$$= -\frac{1}{(\det(A))^2\det(B)} = -\frac{1}{(-1)^2 * 2} = -\frac{1}{2}$$

(b) $\det(((B^{-1}AB)^{2010})^T)$

$$= \det\left((B^{-1}AB)^{2010}\right)$$

$$= \left(\det(B^{-1}AB)\right)^{2010}$$

$$= \left(\det(B^{-1})\ \det(A)\ \det(B)\right)^{2010}$$

$$= \left(\frac{1}{\det(B)}\det(A)\det(B)\right)^{2010}$$

$$= \left(\frac{1}{2} * (-1) * 2\right)^{2010} = 1$$

3. (a) [(4) marks] Show that $W_1 = \{(a, b, c) \in \mathbb{R}^3 \mid a + b = c - b\}$ is a subspace of $\mathbb{R}^3$.

Proof: (1) $\forall u_1 = (a_1, b_1, c_1) \in W_1$, $u_2 = (a_2, b_2, c_2) \in W_1$, we need to show $(a_1, b_1, c_1) + (a_2, b_2, c_2) \in W_1$.

As $(a_1, b_1, c_1) \in W_1$, $(a_2, b_2, c_2) \in W_1$,

$a_1 + b_1 = c_1 - b_1$ and $a_2 + b_2 = c_2 - b_2$

So, $(a_1 + a_2) + (b_1 + b_2) = (c_1 + c_2) - (b_1 + b_2)$

So, $(a_1 + a_2, b_1 + b_2, c_1 + c_2) \in W_1$, i.e, $u_1 + u_2 \in W_1$

(2) $\forall u = (a, b, c) \in W_1$, $\forall k \in R$, we need to show $ku \in W_1$.

Since $u \in W_1$, $a + b = c - b$. So, $ka + kb = kc - kb$.

So, $(ka, kb, kc) \in W_1$; i.e, $ku \in W_1$.

(b) [(5) marks] Show that $W_2 = \{(a, b, c) \in \mathbb{R}^3 \mid (\sin a)b = cb\}$ is not a subspace of $\mathbb{R}^3$.

Here is a counterexample:

Take $u = (\pi, 1, 0) \in W_2$ (as $(\sin\pi) * 1 = 0 * 1$)

Choose $k = \frac{1}{2}$. Then $ku = (\frac{\pi}{2}, \frac{1}{2}, 0)$.

As $\sin(\frac{\pi}{2}) * \frac{1}{2} = \frac{1}{2}$, $0 * \frac{1}{2} = 0$,

$$\sin(\frac{\pi}{2}) * \frac{1}{2} \neq 0 * \frac{1}{2}.$$

So, $ku \notin W_2$, i.e, W_2 is not closed under scalar multiplication. So, W_2 is not a subspace of $\mathbb{R}^3$

4. [(8) marks] For which real values of λ do the following vectors form linearly dependent set in $\mathbb{R}^3$?

$$\mathbf{u}_1 = (\lambda, \frac{1}{2}, \frac{1}{2}), \quad \mathbf{u}_2 = (\frac{1}{2}, \lambda, \frac{1}{2}), \quad \mathbf{u}_3 = (\frac{1}{2}, \frac{1}{2}, \lambda)$$

If u_1, u_2, u_3 are linearly dependent, then

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = 0 \quad \text{—— (1)}$$

has non-trivial solution.

(1) is the following system

$$\begin{cases} \lambda x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3 = 0 \\ \frac{1}{2} x_1 + \lambda x_2 + \frac{1}{2} x_3 = 0 \\ \frac{1}{2} x_1 + \frac{1}{2} x_2 + \lambda x_3 = 0 \end{cases}$$

So, $\begin{vmatrix} \lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \lambda \end{vmatrix} = 0$

i.e., $(\lambda+1)(\lambda-\frac{1}{2})^2 = 0$

So, $\lambda = -1$ or $\lambda = \frac{1}{2}$

5. [(5 + 5) marks] Let $A = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}$.

(a) Solve the equation $\det(\lambda I_3 - A) = 0$.

$$\begin{vmatrix} \lambda & 0 & -4 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda+3 \end{vmatrix} = 0$$

Expanding along 1st row, we get

$$\lambda^2(\lambda+3) + (-4) = 0$$

ie., $(\lambda-1)(\lambda+2)^2 = 0$

So, $\lambda = 1$ or $\lambda = -2$.

(b) Compute $\det(\lambda A^{-1} - I_3)$. (Your answer should be an expression involving the variable λ)

As $A(\lambda A^{-1} - I_3) = \lambda I_3 - A$

$$\det(A(\lambda A^{-1} - I_3)) = \det(\lambda I_3 - A)$$

$$\det(A) \cdot \det(\lambda A^{-1} - I_3) = \det(\lambda I_3 - A)$$

By (a), $\det(\lambda I_3 - A) = (\lambda-1)(\lambda+2)^2$; and

$$\det(A) = \begin{vmatrix} 0 & 0 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{vmatrix} = 4 \quad \text{(Expanding along 1st row)}$$

So, $\det(\lambda A^{-1} - I_3) = \frac{1}{4}(\lambda-1)(\lambda+2)^2$

6. [(5 + 2 + 3) marks)] Let $A = (2, 2, 0)$, $B = (-1, 0, 2)$ and $C = (0, 4, 3)$ be points in $\mathbb{R}^3$.

(a) Find the area of the triangle having vertices A, B and C.

$\overrightarrow{AB} = (-3, -2, 2)$, $\overrightarrow{AC} = (-2, 2, 3)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix} = (-10, 5, -10)$$

The area of the triangle ABC is

$$S = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{(-10)^2 + 5^2 + (-10)^2}$$
$$= \frac{15}{2}$$

(b) Find the length of the side AB.

By (a), $\overrightarrow{AB} = (-3, -2, 2)$. So

$$\|\overrightarrow{AB}\| = \sqrt{(-3)^2 + (-2)^2 + 2^2} = \sqrt{17}$$

(c) Use the results of parts (a) and (b) above to find the length of the altitude h_c from vertex C to the side AB of the triangle having vertices A, B and C.

$$S = \frac{1}{2} \|\overrightarrow{AB}\| \cdot h_c$$

$$\frac{15}{2} = \frac{1}{2} \cdot \sqrt{17}\, h_c$$

$$h_c = \frac{15}{\sqrt{17}} = \frac{15}{17}\sqrt{17}$$

-The end-