MAT 2377 3X (Spring 2011)
Introduction to Simple Linear Regression
Sections 11.1-11.3

§11 Simple Linear Regression

§11.1-11.2 Regression Model

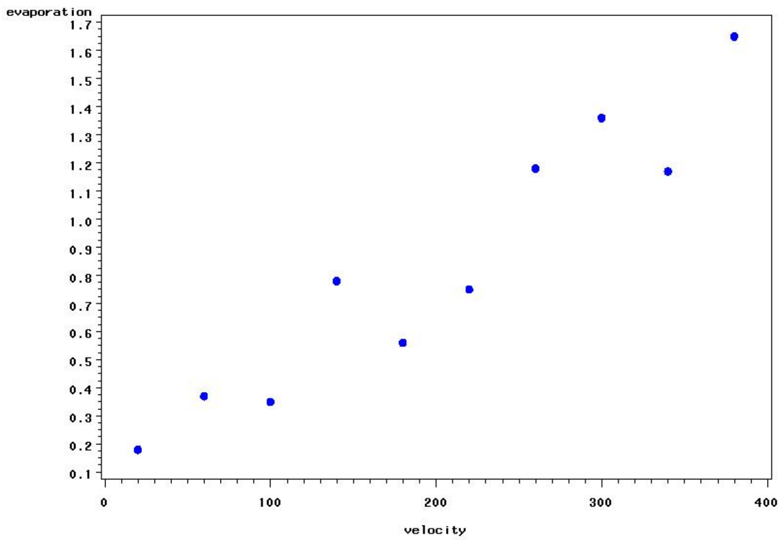
Introduction : We would like to analyze the relationship between two variables. Regression is the study of the relationship between a dependent variable Y and an independent variable X.

Terminology:
Y which is the **dependent variable** is also called the **response variable** and X which is the **independent variable** will be called a **predictor variable**.

Example 1: The following are measurements of the air velocity and evaporation coefficient of burning fuel in an impulse engine:

air velocity cm/sec x	evaporation coefficient mm^2/sec y	air velocity cm/sec x	evaporation coefficient mm^2/sec y
20	0.18	220	0.75
60	0.37	220	0.75
100	0.35	260	1.18
140	0.78	300	1.36
180	0.56	340	1.17
		380	1.65

Here is a scatter diagram of y versus x.



Question: Does there appear to be a linear trend?

We will suppose that there exists a linear statistical relationship between the response Y and the predictor X. We can represent such a relationship with a **simple linear regression** model.

Simple Linear Regression Model is

$$Y = \beta_0 + \beta_1 \, x + \epsilon,$$

where β_0 and β_1 are unknown constants, x is a value taken by the predictor X and ϵ is **random error**.

We will assume that ϵ is a random variable with mean 0 and variance σ^2. That is,

$$E(\epsilon) = 0 \qquad \text{and} \qquad V(\epsilon) = \sigma^2$$

Interpretation of the model:

Given a value x of the predictor variable X, Y is a random variable with mean

$$\mu_{Y|x} = E[Y|x] = \beta_0 + \beta_1 \, x.$$

Terminology:

$$\mu_{Y|x} = \beta_0 + \beta_1 \, x$$

is called the regression line with **intercept** β_0 and **slope** β_1.

Variation: Given a value x of the predictor, the variance of Y is

$$V(Y|x) = V(\epsilon) = \sigma^2.$$

Note: σ^2 is called the variance of the random error.

Terminology: β_0, β_1 and σ^2 are called parameters of the simple linear regression model.

Estimation of the parameters:

Sample : We select a random sample of n paired observations:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

Assuming that the simple linear regression is appropriate, then we can express the observations as follows:

$$y_i = \beta_0 + \beta_1 \, x_i + \epsilon_i, \qquad i = 1, 2, \ldots, n,$$

where ϵ_i represents the ith error (or deviation from the regression line).

We would like to find the line that "best" fits the data. We will use the sum of the squared deviations from the line, that is

$$L = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left[y_i - (\beta_0 + \beta_1 \, x_i) \right]^2,$$

as a measure of distance from the line.

Least-Squares Estimation: This method of estimation consists of minimizing L with respect to β_0 and β_1, by solving

$$\left.\frac{\partial L}{\partial \beta_0}\right|_{\hat{\beta}_0,\hat{\beta}_1} = 0 \qquad \text{and} \qquad \left.\frac{\partial L}{\partial \beta_1}\right|_{\hat{\beta}_0,\hat{\beta}_1} = 0.$$

Simplifying these two equations give a system of linear equations called the **normal equations:**

$$\begin{aligned} n\,\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} y_i \\ \hat{\beta}_0 \sum_{i=1}^{n} x_i + \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n} y_i\, x_i \end{aligned}$$

The solutions of the normal equations are called the **least-squares estimates**.

The least-squares estimate of the slope and intercept are (respectively)

$$\widehat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \quad \text{and} \quad \widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1\,\overline{x},$$

and so the **fitted** or **estimated regression line** is

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1\, x.$$

Notation :

$$\overline{x} = \sum_{i=1}^{n} \frac{x_i}{n} \qquad \text{and} \qquad \overline{y} = \sum_{i=1}^{n} \frac{y_i}{n}$$

$$S_{xx} = \sum_{i=1}^{n} (x_i - \overline{x})^2 = \left(\sum_{i=1}^{n} x_i^2\right) - n\,\overline{x}^2 = \left(\sum_{i=1}^{n} x_i^2\right) - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}$$

$$S_{xy} = \sum_{i=1}^{n} y_i\,(x_i - \overline{x})^2 = \left(\sum_{i=1}^{n} x_i\, y_i\right) - n\,\overline{x}\,\overline{y} = \left(\sum_{i=1}^{n} x_i\, y_i\right) - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n}$$

Exemple 2: Consider the data from Example 1. The $n = 11$ observations yielded
$\sum x_i = 2000$, $\sum y_i = 8.35$,
$\sum x_i^2 = 532,000.0$, $\sum y_i^2 = 9.1097$, and
$\sum x_i\, y_i = 2175.4$.

Suppose that the simple linear regression model is appropriate.

a) Determine the estimated regression line.

b) Estimate the mean evaporation coefficient when the air velocity is $x = 140$.

Estimating the variance of the random error.

Let (x_i, y_i) be the ith pair of observed values in the sample.

We denote the evaluation of the estimated regression line at $x = x_i$, as

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i.$$

Note: $\hat{y}_i$ is called the ith fitted value.

The difference between y_i and $\hat{y}_i$ is called the ith **residual**.

Notation:

$$e_i = y_i - \hat{y}_i.$$

Consider the sum of the squared residuals, that is

$$SS_E = \sum_{i=1}^{n} {e_i}^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2,$$

which is often called the **error sum of squares**.

It can be shown that $E(SS_E) = (n-2)\,\sigma^2$, which implies that

$$\hat{\sigma}^2 = \frac{SS_E}{n-2}$$

is unbiased for estimating σ^2.

Note: It is not necessary to compute each residual since there exist an alternate computational formula for SS_E.

Computational formula for SS_E :

$$SS_E = SS_T - \hat{\beta}_1 \, S_{xy}.$$

where

$$SS_T = S_{yy} = \sum_{i=1}^{n} (y_i - \overline{y})^2 = \left(\sum_{i=1}^{n} y_i^2 \right) - n \, \overline{y}^2 = \left(\sum_{i=1}^{n} y_i^2 \right) - \frac{\left(\sum_{i=1}^{n} y_i \right)^2}{n}.$$

Remark: We sometimes called SS_y the total variation, since it measures the variation among the responses $y_1, \ldots, y_n$.

§11.3 **Properties of the least-squares estimators:**
The least-squares estimator for the slope and the intercept are respectively

$$\widehat{\beta}_1 = \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i - \overline{x}) \, Y_i \quad \text{and} \quad \widehat{\beta}_0 = \overline{Y} - \widehat{\beta}_1 \, \overline{x}.$$

Remark: Both estimators are linear combinations of the independent random variables

$$Y_1, \ldots, Y_n.$$

Thus, we can compute their expectation and variation.

Expectation:

$$E[\widehat{\beta}_1] = \beta_1 \qquad \text{and} \qquad E[\widehat{\beta}_0] = \beta_0$$

Variance:

$$V[\widehat{\beta}_1] = \frac{\sigma^2}{S_{xx}} \qquad \text{and} \qquad V[\widehat{\beta}_0] = \sigma^2 \left[\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}} \right].$$

Remarks:

- The estimators $\widehat{\beta}_1$ and $\widehat{\beta}_0$ are unbiased estimators of β_1 and β_0, respectively.
- The standard deviation of the estimator (that we call standard error) allows us to measure the error in estimation:

$$\sigma_{\widehat{\beta_1}} = \sqrt{V[\widehat{\beta}_1]} = \sqrt{\frac{\sigma^2}{S_{xx}}}$$

and

$$\sigma_{\widehat{\beta_0}} = \sqrt{V[\widehat{\beta}_0]} = \sqrt{\sigma^2 \left[\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}}\right]}.$$

- Since we do not know the true value of σ^2, we can estimate it with

$$\hat{\sigma}^2 = \frac{SS_E}{n-2}.$$

Estimation of the standard errors:

$$\widehat{\sigma}_{\widehat{\beta_1}} = \sqrt{\frac{\hat{\sigma}^2}{S_{xx}}}.$$

$$\widehat{\sigma}_{\widehat{\beta_0}} = \sqrt{\hat{\sigma}^2 \left[\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}}\right]}.$$

Recall:

$$\hat{\sigma}^2 = \frac{SS_E}{n-2} = \frac{S_{yy} - \hat{\beta}_1 \, S_{xy}}{n-2}.$$

Exemple 3: Refer to Example 1 and Example 2.

(a) Compute the 2th residual.
Recall: $x_2 = 60$ and $y_2 = 0.37$.

(b) Give a point estimate for σ^2.

(c) Give the estimated standard error for the estimation of the intercept and also for the estimation of the slope.