Solutions to Test 3 - MATH 1107C - Winter 2010

Version 2

PART I: Multiple choice questions (3 points each)

Choose and circle only one answer. No partial marks here.
No justification is required.

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x,y) = (-y,x)$. Then the inverse of T is:

(**a**)$T^{-1}(x,y) = (-x,y)$ (**b**)$T^{-1}(x,y) = (x,-y)$ (**c**)$T^{-1}(x,y) = (-y,x)$ (**d**)$T^{-1}(x,y) = (y,-x)$

Solution: (**d**)

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1,0) = (1,5)$ and $T(0,1) = (-2,3)$. Then $T(-1,1)$ is:

(**a**)$(-1,8)$ (**b**)$(-1,5)$ (**c**)$(-3,-2)$ (**d**)$(1,7)$

Solution: (**c**)

3. Let A, B and C be $n \times n$ invertible matrices. Consider the following statements:

(i) $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ is an elementary matrix.
(ii) $(ABC)^{-1} = A^{-1}B^{-1}C^{-1}$
(iii) If $A^5 = I$ then $A^{-1} = A^4$.
(iv) If $AB = BC$ then $A = C$.

Which statements are true?

(a) (i) and (ii) only **(b)** (i) and (iii) only **(c)** (ii) and (iv) only **(d)** (iii) and (iv) only

Solution: (**b**)

4. Let A and B be invertible $n \times n$ matrices such that $(4I - 2AX)^{-1} = (B^{-1})^T$. Then:

(a) $X = \frac{1}{2}A^{-1}(4I - B^T)$ **(b)** $X = 2A^{-1}(4I - B^T)$
(c) $X = \frac{1}{2}(4I - B^T)A^{-1}$ **(d)** $X = 2(4I - B^T)A^{-1}$

Solution: (**a**)

PART II: Long answer questions

Show all your work.

5. Let $R_{\pi/2}\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation through $\pi/2$ about the origin, and let $Q_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = x$.

(a) [**3 points**] Find the matrix for $R_{\pi/2}$.

(b) [**3 points**] Find the matrix for Q_1.

(c) [**2 points**] Find the matrix for $R_{\pi/2} \circ Q_1$.

(d) [**2 points**] What kind of geometric transformation is $R_{\pi/2} \circ Q_1$? In other words, is it a rotation (if so, by which angle?) or a reflection (if so, through which line?)? Explain.

Solution: (a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) The matrix obtained in (c), $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, represents a reflection Q through y-axis (or the line $x = 0$) as $Q(1,0) = (-1,0)$ and $Q(0,1) = (0,1)$.

6. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$.

(a) [**9 points**] Find the inverse of A.

(b) [**4 points**] Use A^{-1} to solve the system $AX = B$.

Solution: (a) $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_1 \leftarrow R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] = [I \mid A^{-1}]$

Hence $A^{-1} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

(b) $X = A^{-1}B = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

7. [**10 points**] Find the determinant of $A = \begin{bmatrix} 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 3 & -1 & 2 \\ -2 & 0 & 2 & 0 \end{bmatrix}$ using cofactor expansion.

Solution: By using cofactor expansion along the second column, we obtain:

$$\det A = 3(-1)^{3+2} \det A_{32} = (-3) \cdot \det \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & -1 \\ -2 & 2 & 0 \end{bmatrix} = (-3) \cdot (8) = -24$$