MAT 2377 3X (Spring 2011)

§2.8 **Random Variables**

Definition : Let S be a sample space. A function $X : S \to \mathbb{R}$, that associates a real number $X(s)$ to each outcome s is called a *random variable.*

Notation : The range of the random variable is denoted R_X.

Note : We use upper-case letter (often at the end of the alphabet) to denote random variables, e.g. $X, Y, Z, W, \ldots$

Note : As an outcome is observed and we evaluate the random variables, we obtain an observed value. The observed values are denoted with lower-case letters, e.g. $x, y, z, w, \ldots$

Classification : Let X be a random variable with the range R_X.

1. If the range R_X is a finite set or it is countable, then we say that X is a *discrete* random variable.
2. If the range R_X is an interval (finite or infinite) of real numbers, then X is a *continuous* random variable.

Example 1 : Consider calls to a communication system. Let X be the number of calls between 8 am and 9 am. Let Y be the waiting time between two calls.

(a) Determine an appropriate range for these random variables.

(b) Are these random variables discrete or continuous ?

Events with random variables : We can construct events with random variables. Here are a few examples :

1. Let $A \subseteq \mathbb{R}$, we interpret $\{X \in A\}$ as the following event :

$$\{X \in A\} = \{s \in S \ : \ X(s) \in A\}.$$

2. Let x be a real number, we interpret $\{X = x\}$ as the following event

$$\{X = x\} = \{s \in S \ : \ X(s) = x\}.$$

3. Let x be a real number, we interpret $\{X \leq x\}$ as the following event

$$\{X \leq x\} = \{s \in S \ : \ X(s) \leq x\}.$$

Furthermore,

$$P(X \in A) = P\left(\{s \in S \ : \ X(s) \in A\}\right).$$

Definition : Let X be a random variable. Its *cumulative distribution function* $F : \mathbb{R} \to [0,1]$ is defined as

$$F(x) = P(X \leq x) = P(\{s \in S \ : \ X(s) \leq x\}).$$

Remark :
- We sometimes denote F as F_X. We often use the latter notation when working with more than one random variable.
- We interpret $F(x)$ as the cumulation of mass, where the total mass is 1.

Properties of the cumulative distribution function F :

1. $0 \leq F(x) \leq 1$
2. F is non-decreasing ;
3. $\lim_{x \to -\infty} F(x) = 0.$
4. $\lim_{x \to +\infty} F(x) = 1$
5. For all $a, b \in \mathbb{R}$, such that $a < b$, we have

$$\begin{aligned} F(b) - F(a) &= P(a < X \leq b) \\ &= P\left(\{s \in S \ : \ a < X(s) \leq b\}\right) \end{aligned}$$

§3.1-3.3 **Discrete Random Variables**

Definition : Let X be a random variable with the range R_X. Its *probability mass function* is a function f (sometimes denoted f_X) such that

$$f(x) = P(X = x), \quad \text{pour } x \in R_X.$$

Note : We consider $f(x)$ as a mass.

Properties of f :

1. $0 \leq f(x) \leq 1$
2. $\sum_{x \in R_X} f(x) = 1$
3. **[Computational Property]** Soit $A \subseteq \mathbb{R}$, alors

$$P(X \in A) = \sum_{x \in R_X \cap A} f(x)$$

En particulier, Soit F la fonction de répartition de X, alors

$$F(x) = P(X \leq x) = \sum_{y \in R_X \; : \; y \leq x} f(y).$$

Remark : A function $f : \mathbb{R} \to R_X$, where $R_X \subseteq \mathbb{R}$, that satisfies the above properties (1) and (2) is called a **probability mass function**.

Example 2 : An electronic device contains two components such that each works independently of the other. The probability that the first component is defective is 0.1 and the probability that the second component is defective is 0.2. Let X be the number of defective components in the device.

(a) Give the range of X ?

(b) Determine the p.m.f. of X.

(c) Determine the c.d.f. of X.

(d) What is the probability that there will be at least one defective component ?

(e) What is the probability that there are more than 2 defective components ?

(f) Determine $P(-0.2 < X \leq 1.5)$?

§3.4 **Expected value, Mean and Variance**

Definition : Let X be a discrete random variable with range R_X et probability mass function f. Its *expected value* is $h(X)$ is defined as

$$E[h(X)] = \sum_{x \in R_X} h(x)\, f(x).$$

Remark :
– An expected value is a weighted average.
– We say that $E[h(X)]$ is an **expected value** for the following reason : If we repeat the random experiment a large number of times, then, on average, the observed value of $h(X)$ should be approximately equal to $E[h(X)]$.

Definition : Let X be a discrete random variable with range R_X and probability mass function f. Its mean (also called its expected value) is

$$\mu_X = \mu = E[X] = \sum_{x \in R_X} x\, f(x).$$

Its *variance* is

$$\sigma_X^2 = \sigma^2 = V(X) = E[(X-\mu)^2] = \sum_{x \in R_X} (x-\mu)^2\, f(x).$$

Its *standard deviation* is

$$\sigma_X = \sigma = \sqrt{V(X)}.$$

Alternative Formula for the variance : The following is a alternative formula that can be used to compute the variance. It is more efficient compared to the definition.

$$\sigma^2 = V(X) = E[X^2] - \mu^2 = \left(\sum_{x \in R_X} x^2\, f(x) \right) - \mu^2.$$

Remarks :

- The mean is a centre of mass of the distribution of probability masses. So we say that it is a measure of **central tendency**.
- We also say that the mean is a measure of **location**.
- The standard deviation is a measure of **dispersion** and of **variability**.
- We compute the mean and the variance with the p.m.f. f (and **not** with the c.d.f. F).

Example 3 : Compute the mean and the variance of the random variable X from Example X.

Example 4 : Let X and Y be random variable with the following probability mass functions :

x	$f_X(x)$
-2	$1/5$
-1	$1/5$
0	$1/5$
1	$1/5$
2	$1/5$

y	$f_Y(y)$
-4	$1/5$
-2	$1/5$
0	$1/5$
2	$1/5$
4	$1/5$

Compare σ_X and σ_Y.