Probability - Part III

§2.1.4 **Counting Techniques**

Tree Diagram: If the experiment can be described as a sequence of k steps, then the sample space can be illustrated with a tree. A path in this tree represents an outcome.

Example 12: We select an operator, say Arthur, Beatrice or Celina, and we select one of two machines. Construct the tree diagram and list the possible outcomes.

Remark: It is easy to build a tree, however it can become very large very fast. We will need a more efficient counting technique.

Multiplication Principle: If we can describe an experiment as a sequence of k steps, and that the number of different ways to accomplish the ith task is n_i, then

$$\#\text{ of outcomes } = n_1 \times n_2 \times \cdots \times n_k.$$

Example 13: Consider Example 12. We choose an operator, a machine, then a unit to test among 5 units. In how many different ways can we accomplish this task?

n Factorial

Definition:

Let n be a non-negative integer, that is $n = 0, 1, 2, \ldots$. We define **n factorial** as follows

$$n! = \begin{cases} n\,(n-1)\,(n-2) \times \cdots \times 1, & \text{for } n \geq 1 \\ 1, & \text{for } n = 0 \end{cases}$$

Example 14: $0! = 1;$ $1! = 1;$ $2! = 2 \cdot 1 = 2;$ $3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24;$ $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Many experiments can be perceived as an arrangement or a selection of objects. Since we can encounter such experiments in practice, we will give them a name.

Definition: Consider n distinct objects. An arrangement of r objects is called a **permutation**.

Remark: For a permutation order is important.

Notation:

$P_r^n =$# of different permutations of size r chosen from n objects.

Definition: Consider n distinct objects. A selection of r objects is called a **combination**.

Remark: For a combination order is not important.

Notation:

$C_r^n = \begin{pmatrix} n \\ r \end{pmatrix} =$ # of different combinations of size r chosen among n objects.

Example 15: Consider a set of objects $\{a, b, c, d\}$. Here is a list of all the permutations of these 4 objects of size 2:

$$ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc.$$

Thus, $P_2^4 = 12$

Here is a list of all the combinations of these 4 objects of size 2:

$$ab, ac, ad, bc, bd, cd.$$

Thus, $\left(\begin{array}{c} 4 \\ 2 \end{array} \right) = 6$

Note: For a combination, it is the selection of the objects that is important and not the order. This means that ab and ba are the same combination.

Formulae:

$$P_r^n = n\,(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

et

$$\left(\begin{array}{c} n \\ r \end{array} \right) = C_r^n = \frac{n!}{r!\,(n-r)!}$$

Example 16: A control mechanism requires the placement of 3 chips. Assume that we have access to 5 different types of chips. In how different ways can we assemble the control mechanism by placing 3 chips in the controller?

Example 17: Consider a lot of 50 articles such that 3 are defective. We select 5 articles at random.
(a) In how many ways different ways can we choose the 5 articles?
(b) In how many different ways can we choose 5 articles with the condition that there is exactly 1 defective article.

Example 18: A machine produces three types of holes: small, medium and large. Suppose that we can program the machine to produce a sequence of operations. In how many different ways can we program the machine if we want 2 small, 2 medium and 3 large holes?