

Probability - Part II

Conditional Probability

Definition: If $P(B) > 0$, the *conditional probability* that A will occur given that B occurs is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Remark:

1. We can show that the function $P(\cdot|B)$ satisfies the three probability axioms. Thus, the consequences of the axioms also apply to conditional probabilities. For example,
 - (a) $P(\emptyset|B) = 0$.
 - (b) If $A \subseteq C$, then $P(A|B) \leq P(C|B)$.
 - (c) $0 \leq P(A|B) \leq 1$, for all events A .
 - (d) $P(A'|B) = 1 - P(A|B)$, for all events A .
2. For the equally likely model, the definition of a conditional probability is equivalent to

$$P(A|B) = \frac{N(A \cap B)}{N(B)}.$$

For the latter approach, we interpret $P(A|B)$ as the proportion of outcomes from B that satisfy criteria for A .

Remark: We can also interpret the context for which we use a conditional probability as a change in the experimental conditions.

Example 3: Consider a box of 20 light bulbs where in 2 bulbs are defective. Suppose that we choose two bulbs without replacement. What is the probability that the second bulb is defective given that the first was not defective?

Multiplication Rule:

$$P(A \cap B) = P(B | A)P(A) = P(A | B)P(B)$$

Example 4: Consider a box of 20 light bulbs where in 2 bulbs are defective. Suppose that we choose two bulbs without replacement. What is the probability that both bulbs are not defective?

Total Probability Rule:

$$\begin{aligned}P(B) &= P(A \cap B) + P(A' \cap B) \\ &= P(B | A)P(A) + P(B | A')P(A')\end{aligned}$$

If E_1, \dots, E_k are mutually exclusive and exhaustive (i.e. $E_i \cap E_j = \emptyset$ if $i \neq j$ and $E_1 \cup \dots \cup E_k = S$), then for any event B , we have

$$\begin{aligned}P(B) &= P(B \cap E_1) + \dots + P(B \cap E_k) \\ &= P(B | E_1)P(E_1) + \dots + P(B | E_k)P(E_k)\end{aligned}$$

Bayes' Theorem: If E_1, \dots, E_k are mutually exclusive and exhaustive (i.e. $E_i \cap E_j = \emptyset$ if $i \neq j$ and $E_1 \cup \dots \cup E_k = S$), then for any event B and any i ,

$$P(E_i | B) = \frac{P(B | E_i)P(E_i)}{P(B | E_1)P(E_1) + \dots + P(B | E_k)P(E_k)}$$

Example 5: Samples taken from a river near an industrial plant are tested for toxic levels of lead and mercury: 32% contain toxic levels of mercury, 16% contain toxic levels of lead, and 38% contain toxic levels of at least one of the two substances.

1. A sample chosen at random contains toxic levels of mercury. What is the probability that it also contains toxic levels of lead?
2. It is known that 45% of the samples containing toxic levels of mercury also contain toxic levels of arsenic. What is that probability that a sample contains toxic levels of both mercury and arsenic?
3. From these samples that do not contain toxic levels of mercury, only 2% contain toxic levels of arsenic. What is the probability that the sample contains toxic levels of arsenic?

Example 7: Nissan sold three models of cars in 1999 : Sentras, Maximas and Pathfinders. Of the vehicles sold, 50% were Sentras, 30% were Maximas and 20% were Pathfinders. For that same year, 12% of the Sentras, 15% of the Maximas and 25% of the Pathfinders had a problem in the ignition system.

1. What is the probability that a Nissan sold in 1999 had a problem with the ignition system?
2. A 1999 Nissan has problems with the ignition system, what is the probability that it is a Pathfinder?

Definition: Two events A and B are **independent** if at least of the following statements is true:

1. $P(B | A) = P(B)$
2. $P(A | B) = P(A)$
3. $P(A \cap B) = P(A)P(B)$

Note: The three statements are equivalent when $P(A) > 0$ and $P(B) > 0$.

Definition: The events E_1, \dots, E_k are independent, if for all subcollections $E_{i_1}, E_{i_2}, \dots, E_{i_j}$,

$$P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_j}) = P(E_{i_1}) \times P(E_{i_2}) \times \dots \times P(E_{i_j})$$

Remark: If we have a collection of independent events and we substitute this event by its complement, then the new collection is also a collection of events.

For example: If A and B are independent, then

- A and B' are independent;
- A' and B are independent;
- A' and B' are independent.

Example 8: Consider an experiment with equally likely results such that $N(S) = 20$, $N(A) = 5$, $N(B) = 10$ and $N(A \cap B) = 3$.

Are A and B independent?

Example 9: Consider an experiment with equally likely results such that $N(S) = 20$, $N(A) = 5$, $N(B) = 12$ et $N(A \cap B) = 3$.

Are A and B independent?

Remark: Do not confuse the concept of independent events with the concept of mutually exclusive events. Both concepts imply different things.

- If A and B are **independent events**, then

$$P(A \cap B) = P(A) P(B)$$

and

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A) P(B) \end{aligned}$$

- If A and B are **mutually exclusive events**, then

$$P(A \cap B) = P(\emptyset) = 0$$

et

$$P(A \cup B) = P(A) + P(B).$$

Example 10: Suppose that an electronic component will work is 0.90. Consider 5 independent components.

- (a) What is the probability that all 5 will work?
- (b) Compute the probability that at least 1 will work?
- (c) What is the probability that only the third component will work?

Example 11: Consider the following circuit. We say that it is functional if there is a path of functional components from left to right. The probability that the component is functional is illustrated. Suppose components operate independently of each other. What is the probability that the circuit is functional?

