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Time 2 hours
Marks Each of the six questions are of approximately equal weight.

1. (a) Evaluate
$$\binom{\frac{4}{3}}{3}.$$
(b) Determine $[x^5](1-2x)^{-3}$.
(c) Determine $[x^n](1+3x)^m(1-x^2)^{-5}$.
(d) Prove that
$$\left|\sum_{j \text{ even}} 2^j\binom{n}{j} - \sum_{j \text{ odd}} 2^j\binom{n}{j}\right| = 1.$$

2. (a) Let c_n be the number of compositions of n, $n \geq 0$, with an odd number of parts, and in which all parts are odd positive integers. Prove that
$$\sum_{n\geq 0} c_n x^n = \frac{x-x^3}{1-3x^2+x^4}.$$
(b) From part (a), determine a linear recurrence equation for c_n, together with initial conditions that uniquely determine $\{c_n\}_{n\geq 0}$.
(c) For fixed positive integers n and k, determine the number of compositions of n into $2k$ parts where the 2nd, 4th, ..., $2k$th parts are either 1 or 2, and the remaining parts are any odd, positive integer.

3. (a) Let $\mathcal{R} = \{\epsilon, 0, 01\}\{0, 1, 01\}$. Are the elements of $\mathcal{R}$ uniquely created in this decomposition? Explain.
(b) Let $\mathcal{S} = \{0, 1, 01\}\{\epsilon, 0, 01\}$. Are the elements of $\mathcal{S}$ uniquely created in this decomposition? Explain.
(c) Let $\mathcal{T} = \{\epsilon, 11\}(\{0\}\{00\}^*\{11\}\{11\})^*\{\epsilon, 00\}$. Find the generating function for $\mathcal{T}$ with respect to length.
(d) Let $\mathcal{A}$ be the set of $\{0,1\}$-strings in which every substring of length three has at least one 0 and at least one 1. Prove that the generating fuction for $\mathcal{A}$ with respect to length is given by
$$\Phi_{\mathcal{A}}(x) = \frac{1+x+x^2}{1-x-x^2}.$$

4. (a) Verify that n^2+5n is a particular solution to the nonhomogeneous recurrence equation
$$b^n - 2b_{n-1} - 5b_{n-2} + 6b_{n-3} = 2 - 12n, \quad n \geq 3.$$
(b) Solve the recurrence equation given in part (a), subject to initial conditions $b_0 = 2$, $b_1 = 0$, $b_2 = 14$.
(c) Determine an asymptotic form for b_n in part (b).

5. (a) Give the definition of a tree.
(b) Draw three nonisomorphic trees on 5 verticies.
(c) Determine the smallest number of verticies r in a tree having 5 verticies of degree 2, 2 verticies of degree 3, and 2 verticies of degree 5. Justify your answer by proving that every such tree has at least r verticies, and by giving an example of a tree with exactly r verticies.
(d) Prove that a connected graph in which every vertex has degree greater than or equal to 2 must have a cycle.

6. (a) Define a graph G_n, $n \geq 1$, in the following way. Let the verticies of G_n be the $\{0,1\}$-strings of length $2n$ that have exactly n 1s and exactly n 0s. Two verticies are adjacent if and only if they differ in exactly 2 positions. Draw G_1 and G_2.
(b) Determine the number of verticies and edges in G_n, $n \geq 1$.
(c) Without drawing G_4, find a graph in G_4 from vertex $u = 00001111$ to vertex $v = 11110000$.
(d) Determine the values of $n \geq 1$ for which G_n is connected.