Name: **Student Id:** **Signature:**

Instruction: Only non-programmable calculators are allowed. Part A does not require justification. For questions in part B, give justified answers in good handwriting, unjustified answers will not get credit.

[8] **Part A** No justification is required for the multiple choice questions in this part. Each of the following questions has exactly one correct answer. **Circle** the correct answer. 2 marks for each question.

A1. Let A and B be 3×3 matrices with $|A| = -1$ and $|B| = 10$. Then $|2AB^T|$ is equal to

(a) 20 (b) -20 (c) 40 (d) -80

A2. The determinant of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix}$ is equal to

(a) 6 (b) 0 (c) -2 (d) 2

A3. Let $A = \begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}$. Which of the following vectors is an eigenvector of A?

(a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$

A4. If $A = PDP^{-1}$, then

(a) $A^5 = P^5D^5P^{-5}$ (b) $A^5 = PD^5P^{-1}$ (c) $A^5 = P^5DP^{-5}$ (d) $A^5 = P^{-1}D^5P$

Part B Give justified answers in good handwriting, unjustified answers will not get credit.

[6]B1. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & -2 \end{bmatrix}$.

(i) Use the cofactor expansion across row 2 to compute $|A|$.

(ii) Use the cofactor expansion across column 3 to compute $|A|$.

Solution (i)

$|A| = (1)\begin{vmatrix} 2 & 0 \\ 4 & -2 \end{vmatrix} = (1)(-4) = -4.$ 2 marks for the first equality, and 1 mark for the last equality

(ii)

$|A| = (-2)\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = (-2)(2) = -4.$ 2 marks for the first equality, and 1 mark for the last equality

[10]B2. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(i) Find all the eigenvalues of A.

(ii) For each eigenvalue found in (i), find a corresponding eigenvector.

(iii) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution (i) The matrix is triangular, and hence the eigenvalues are 1 and 2 (its diagonal entries).
2 marks
(ii) To find an eigenvector corresponding to the eigenvalue 1, we solve the equation $(A - I)\mathbf{x} = \mathbf{0}$.
The augmented matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Hence x_1 is a free variable, and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence $\mathbf{v}_1 = (1, 0)$ is an eigenvector corresponding to the eigenvalue 1. 3 marks
To find an eigenvector corresponding to the eigenvalue 2, we solve the equation $(A - 2I)\mathbf{x} = \mathbf{0}$. The augmented matrix is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence x_2 is a free variable, and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence $\mathbf{v}_2 = (1, 1)$ is an eigenvector corresponding to the eigenvalue 2. 3 marks
(iii) We have $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

2 marks