Solutions to Test 3 - MATH 1107C - Winter 2010

Version 1

PART I: Multiple choice questions (3 points each)
Choose and circle only one answer. No partial marks here.
No justification is required.

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x,y) = (y,-x)$. Then the inverse of T is:

(a) $T^{-1}(x,y) = (-x,y)$ **(b)** $T^{-1}(x,y) = (x,-y)$ **(c)** $T^{-1}(x,y) = (-y,x)$ **(d)** $T^{-1}(x,y) = (-y,-x)$

Solution: (c)

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1,0) = (2,4)$ and $T(0,1) = (-1,3)$. Then $T(1,-1)$ is:

(a) $(1,-3)$ **(b)** $(5,5)$ **(c)** $(3,1)$ **(d)** $(1,7)$

Solution: (c)

3. Let A, B and C be $n \times n$ invertible matrices. Consider the following statements:
(i) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ is an elementary matrix.
(ii) If $A^3 = I$ then $A^{-1} = A^2$.
(iii) $(ABC)^{-1} = A^{-1}B^{-1}C^{-1}$
(iv) If $AB = CA$ then $B = C$.
Which statements are true?

(a) (i) and (ii) only **(b)** (i) and (iv) only **(c)** (ii) and (iii) only **(d)** (iii) and (iv) only

Solution: (a)

4. Let A and B be invertible $n \times n$ matrices such that $(5I - 3AX)^{-1} = (B^{-1})^T$. Then:

(a) $X = 3A^{-1}(5I - B^T)$ **(b)** $X = \frac{1}{3}A^{-1}(5I - B^T)$
(c) $X = 3(5I - B^T)A^{-1}$ **(d)** $X = \frac{1}{3}(5I - B^T)A^{-1}$

Solution: (b)

PART II: Long answer questions
Show all your work.

5. Let $R_{\pi/2}\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation through $\pi/2$ about the origin, and let $Q_{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = -x$.

(a) **[3 points]** Find the matrix for $R_{\pi/2}$.

(b) **[3 points]** Find the matrix for Q_{-1}.

(c) **[2 points]** Find the matrix for $R_{\pi/2} \circ Q_{-1}$.

(d) **[2 points]** What kind of geometric transformation is $R_{\pi/2} \circ Q_{-1}$? In other words, is it a rotation (if so, by which angle?) or a reflection (if so, through which line?)? Explain.

Solution: **(a)** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) The matrix obtained in (c), $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, represents a reflection Q_0 through x-axis (or the line $y = 0$) as $Q_0(1, 0) = (1, 0)$ and $Q_0(0, 1) = (0, -1)$.

6. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(a) **[9 points]** Find the inverse of A.

(b) **[4 points]** Use A^{-1} to solve the system $AX = B$.

Solution: **(a)** $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_1 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] = [I \mid A^{-1}]$

Hence $A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

(b) $X = A^{-1}B = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$

7. **[10 points]** Find the determinant of $A = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 2 \\ -2 & 1 & 2 & 0 \end{bmatrix}$ using cofactor expansion.

Solution: By using cofactor expansion along the fourth column, we obtain:

$$\det A = 2(-1)^{3+4} \det A_{34} = (-2) \cdot \det \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & 0 \\ -2 & 1 & 2 \end{bmatrix} = (-2) \cdot 5 = -10.$$