Concordia University

Faculty of Engineering and Computer Science

Midterm Exam ENGR 233

20/20 ✓

Thursday, October 23, 2014. *1:15 – 2:30 P.M.*

Note: Calculators are not allowed.

Problem 1. [4 points]

Find a vector function $\bar{r}(t)$ that satisfies the following conditions:

$\bar{r}'(t) = 6\hat{i} + 6t\hat{j} + 3t^2\hat{k}; \quad \bar{r}(0) = \hat{i} - 2\hat{j} + \hat{k}$

[4 points] Problem 2. [4 points]

The motion of a particle in space is described by.

$\bar{r}(t) = b\cos t\,\hat{i} + b\sin t\,\hat{j} + ct\,\hat{k};\ t \geq 0$. Compute $\|\bar{v}(t)\|$

Problem 3. [4 points]

Find the tangential and normal components of the acceleration at any t, if the displacement vector is given by

$\bar{r}(t) = \hat{i} + t\hat{j} + t^2\hat{k}$

Problem 4. [4 points]

Using chain rule, find the indicated partial derivatives for the functions given below:

$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y},\ \text{where } z = u^2\cos 4v,\ u = x^2y^3,\ v = x^3 + y^3,$

Problem 5. [4 points]

The temperature at a point (x,y) on a rectangular metal plate is given by

$T(x, y) = 100 - 2x^2 - y^2$. Find the path a heat seeking particle will take, starting at (3,4) as it moves in the direction in which the temperature increases most rapidly.

$\boxed{1}$ $\vec{r}'(t) = 6\vec{i} + 6t\vec{j} + 3t^2\vec{k}$

Integrating

$\vec{r}(t) = (6t + c_1)\vec{i} + (3t^2 + c_2)\vec{j} + (t^3 + c_3)\vec{k}$

$\vec{r}(0) = \vec{i} - 2\vec{j} + \vec{k}$

therefore, $6 \times 0 + c_1 = 1$

$3 \times 0^2 + c_2 = -2$

$0^3 + c_3 = 1$

$\vec{r}(t) = (6t + 1)\vec{i} + (3t^2 - 2)\vec{j} + (t^3 + 1)\vec{k}$ ✓ ✓ ✓

4/4

$\boxed{2}$ $\vec{r}(t) = b\cos t\,\vec{i} + b\sin t\,\vec{j} + ct\,\vec{k}$

$\vec{v}(t) = -b\sin t\,\vec{i} + b\cos t\,\vec{j} + c\,\vec{k}$

$\|\vec{v}\| = \sqrt{(-b\sin t)^2 + (b\cos t)^2 + c^2}$

$= \sqrt{b^2\sin^2 t + b^2\cos^2 t + c^2}$

$= \sqrt{b^2(\sin^2 t + \cos^2 t) + c^2}$ ✓

$\|\vec{v}(t)\| = \sqrt{b^2 + c^2}$ ✓

4/4

$\boxed{3}$ $\vec{r}(t) = \vec{i} + t\vec{j} + t^2\vec{k}$ $\qquad a_n = \frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|}$ $\qquad a_t = \frac{\|\vec{a}\times\vec{v}\|}{\|\vec{v}\|}$

$\vec{v}(t) = \vec{j} + 2t\vec{k}$ $\qquad \|\vec{v}(t)\| = \sqrt{1 + 4t^2}$

$\vec{a}(t) = 2\vec{k}$

- $a_n = \frac{2\vec{k}\cdot(\vec{j} + 2t\vec{k})}{\sqrt{1 + 4t^2}}$

$= \frac{4t}{\sqrt{1 + 4t^2}}$ ✓

- $a_t = \frac{\|2\vec{k} \times (\vec{j} + 2t\vec{k})\|}{\sqrt{1 + 4t^2}}$

$2\vec{k} \times (\vec{j} + 2t\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 2 \\ 0 & 1 & 2t \end{vmatrix} = -2\vec{i}$ ✓

$a_t = \frac{2}{\sqrt{1 + 4t^2}}$ ✓

4/4

$\boxed{4}$ $z = u^2 \cos(4v)$ $u = x^2 y^3$ $v = x^3 + y^3$

- $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}$

$= 2u\cos(4v) \times 2xy^3 \quad - 4u^2 \sin(4v) \times 3x^2$

$= 4u\cos(4v)\, xy^3 - 12u^2 \sin(4v)\, x^2$ ✓

- $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}$

$= 2u\cos(4v) \times 3x^2y^2 - 4u^2 \sin(4v) \times 3y^2$

$= 6u\cos(4v)\, x^2y^2 - 12u^2 \sin(4v)\, y^2$ ✓

$\dfrac{4}{4}$

$\boxed{5}$ • $T(x,y) = 100 - 2x^2 - y^2$ $(3,4)$

$\nabla T(x,y) = \dfrac{\partial T}{\partial x}\vec{i} + \dfrac{\partial T}{\partial y}\vec{j}$

$\nabla T(x,y) = -4x\,\vec{i} - 2y\,\vec{j}$ ✓

$(\nabla T(3,4) = -12\vec{i} - 8\vec{j})$

$\dfrac{4}{4}$

- And $\begin{cases} x'(t) \\ y'(t) \end{cases} = \nabla T(x,y)$

So, $\dfrac{dx}{dt} = -4x$

$\dfrac{dy}{dt} = -2y$

By separation of variable,

- $\dfrac{dx}{x} = -4\,dt$ • $\dfrac{dy}{y} = -2\,dt$

$\ln x = -4t + C1$ $\ln y = -2t + C2$

at $t=0$, $x=3$, $y=4$

$\Rightarrow \ln 3 = C1 \Rightarrow \ln 4 = C2$

$\ln x = -4t + \ln 3$

$\ln y = -2t + \ln 4$

$-4t = \ln x - \ln 3$

$t = \dfrac{\ln 3 - \ln x}{4}$

$t = \dfrac{\ln\left(\frac{3}{x}\right)}{4}$

$\Rightarrow \ln y = -\dfrac{1}{2}\ln\dfrac{3}{x} + \ln 4$

$\Leftrightarrow 2\ln y = 2\ln 4 - \ln\dfrac{3}{x}$

$\Leftrightarrow \ln y^2 = \ln 16 - \ln\dfrac{3}{x}$

$\Leftrightarrow \ln y^2 = \ln\left(\dfrac{16}{\frac{3}{x}}\right)$

$\Leftrightarrow \ln y^2 = \ln\dfrac{16x}{3}$

$\Leftrightarrow y^2 = \dfrac{16x}{3}$

$\Leftrightarrow 3y^2 = 16x$

$\boxed{x = \dfrac{3}{16}y^2}$ ✓