MAT2384: Ordinary Differential Equations and Numerical Methods

Dr. Robert Smith?

Contents

General Introduction

0.1 Basic Definitions

0.1.1 Differential equation

A differential equation is an equation involving an unknown function and some of its derivatives.

0.1.2 Examples

$$\frac{dy}{dt} = 10y^2 + \sin(t)$$
$$y'' + 10y'y = 0$$

Here y is really $y(t)$.It's easy to spot which are the variables and which are the unknown constants: variables have derivatives.

A derivative is a rate of change, so functions are used because they change (with respect to time, or space or whatever).

0.1.3 Order

The order of the differential equation is the order of the highest derivative appearing in the equation.

0.1.4 Examples

$$\frac{dy}{dt} = 10y^2 + \sin(t)$$

has order 1 while

$$y'' + 10y'y = 0$$

has order 2.

0.1.5 Linear equations

A linear equation is an equation in which the unknown function $y(t)$ and its derivatives appear by themselves. For example,

$$\frac{d^2y}{dt^2} = 10\left(\frac{dy}{dt}\right)^2 + \sin(t)$$

is nonlinear while

$$y'' + 10y' + 2y = 0$$

is linear.

Similarly, the equation

$$y'' + (\sin t)y' + (\cos^2 t)y = 6$$

is linear, because the function y and its derivatives are affected only by addition and scalar (with respect to y) multiplication.

0.1.6 Existence and uniqueness

Existence and uniqueness are important questions. Does a solution for an equation exist? An existence theorem might tell us if it does. Is the solution unique? Some equations may have no solutions at all and some may have infinitely many. Some equations have infinitely many solutions but only one solution that satisfies certain conditions.

0.1.7 Initial value problem

If we are given a differential equation and all the required conditions that the solution must satisfy at a single point then we have an initial value problem. For example,

$$y'' + y' + y = 1 \text{ with } y(0) = 1 \text{ and } y'(0) = 3$$

Many functions may satisfy the DE and not the initial conditions. For example, $y(t) = 1$ satisfies the DE and one condition but not the other - so it doesn't satisfy the IVP.

0.1.8 Two-point boundary value problem

If we are given a differential equation but the conditions are at two different values of t, we have a two-point boundary value problem. For example,

$$y'' + y' + y = 1 \text{ with } y(0) = 1 \text{ and } y(1) = 3$$

0.1.9 General solutions

If we are not given any conditions then we can only find a general solution that will usually contain one or more unknowns.

Example.

$$y'' + y = 0$$

has the solutions

$$y = A\sin(t) + B\cos(t)$$

where A and B are arbitrary constants.

This allows us to add in initial or boundary conditions later.

0.2 Modelling with ODEs

0.2.1 Introduction

An important aspect of the study of ordinary differential equations is the use of such equations in the study of problems in diverse areas such as physical and biological sciences and finance.
The basic idea is that we build a mathematical model of the problem which lets us investigate relevant characteristics while being simple enough to study and analyze.
Anything that changes in some way can be modelled by DEs.
We look at some simple models.

0.3 Growth and decay of populations

Let the population of cod in the North Atlantic at a time t be $C(t)$. Suppose that we assume that the change in population at a time t is proportional to the population itself. Then

$$\frac{dC}{dt} = kC$$

where k is the constant of proportionality. If k is positive then the population will increase if k is negative then the population will decrease.
This is an example of a separable equation. We rearrange the terms to get

$$\frac{dC}{C} = kdt$$

and then integrate both sides to get

$$C(t) = C_0 e^{kt}$$

where $C_0 = C(0)$. This simple model suggests that the cod population will grow (or decrease) exponentially with time. This neglects the fact that a population cannot grow beyond limits set by availability of resources such as food. Let's see how we can incorporate this idea onto a model.
Suppose that we can set a limit to the number of cod based on information about food sources. Let this limit be C_{max}. Then a new model might be

$$\frac{dC}{dt} = kC\left(1 - \frac{C}{C_{max}}\right)$$

We see that this equation is nonlinear. For C small compared to C_{max} we have approximately our original linear equation. As C approaches C_{max} the time derivative goes to zero and so the population levels off at C_{max}.
The solution is of the form

$$C(t) = \frac{C_{max}C_0}{C_0 + (C_{max} - C_0)e^{-kt}}$$

where $C_0 = C(0)$.
A separable equation is one where we can separate the variables, in this case putting all the Cs on one side (including dC) and the t's on the other. The constant k could go on either side.
We'll see later how to get this solution. For now, try differentiating and putting it into the original equation with the initial condition to verify that it is the solution.

0.3.1 Compound interest

Example. If we invest 100 dollars at 5 percent interest, after one year we have a sum of 100*1.05 and after n years we have a sum of

$$100 \times (1.05)^n$$

In general if we invest an amount D_0 at an interest rate r compounded m times a year for t years we get

$$D_0(1 + r/m)^{mt}$$

How much do we have if we compound continuously?

If we let the number of times that the interest is compounded increase indefinitely, i.e. $m \to \infty$, we get continuous compounding and we see that the capital $D(t)$ after t years is

$$D(t) = \lim_{m\to\infty} D_0(1 + r/m)^{mt}$$

Take the limit:

$$\begin{aligned}
\ln D(t) &= \ln D_0 + \lim_{m\to\infty} mt\ln(1 + r/m) \\
&= \ln D_0 + \lim_{m\to\infty} \frac{\ln(1 + r/m)}{1/(mt)} = \ln D_0 + \frac{\infty}{\infty} \\
&= \ln D_0 + \lim_{m\to\infty} \frac{\frac{1}{1+r/m} \cdot \frac{-r}{m^2}}{-\frac{1}{m^2 t}} \qquad \text{(L'Hôpital's Rule)} \\
&= \ln D_0 + \lim_{m\to\infty} \frac{rt}{1 + r/m} \\
&= \ln D_0 + rt.
\end{aligned}$$

Thus

$$D(t) = D_0 e^{rt}$$

This is the same as for our growth and decay example, so the problem of continuously compounded interest can be modelled by the differential equation

$$D' = rD \qquad \text{with} \qquad D(0) = D_0.$$

0.3.2 Radioactive Decay

Some elements decay at a rate which depends on the amount of material present. The more of the element that is present the faster the decay. Thus if $Q(t)$ is the amount of the element present at time t, the rate of decay could be modelled as

$$\frac{dQ}{dt} = -rQ$$

where r is a positive constant that differs from material to material. Thus if $Q(0) = Q_0$ our previous results give us

$$Q(t) = Q_0 e^{-rt}$$

The half-life is the amount of time taken for the sample to decay to half of what it was originally. If T is the half-life then

$$Q(t) = \frac{1}{2}Q_0 = Q_0 e^{-rT}$$

Solve for r in terms of T to get

$$r = -\frac{\log 0.5}{T} = \frac{0.6931}{T}$$

and so we can also express the decay law as

$$Q(t) = Q_0 e^{-0.6931/Tt}$$

1 First order differential equations

1.1 First order equations

1.1.1 Form

A general form of these equations is

$$\frac{dy}{dt} = f(y,t)$$

where we will assume that $f(y,t)$ is a reasonably well-behaved function of both y and t.

1.1.2 Conditions

Usually, we are also given a condition at some value of t, such as

$$y(0) = y_0$$

where y_0 is some constant.

1.1.3 Solutions

We do not have a procedure for solving the general first order equation for arbitrary f. However, there are solutions for separable and linear equations which we consider next.

1.2 Separable equations

1.2.1 Form

If the equation

$$\frac{dy}{dt} = f(y,t)$$

can be expressed as a product of a function of y times a function of t, that is, in the form

$$\frac{dy}{dt} = Y(y)T(t)$$

we call the equation separable.

1.2.2 Solutions

The expression for a separable equation can be manipulated to give

$$\frac{dy}{dt} = Y(y)T(t)$$
$$\frac{dy}{Y(y)} = T(t)dt$$

and then integrated on both sides to give

$$\int \frac{dy}{Y(y)} = \int T(t)dt + K$$

where K is an arbitrary constant which will be evaluated by applying the condition at $t = 0$. Of course, the two integrations may be difficult or impossible.
The constant of integration K is absolutely vital here. Sometimes it does turn out to be zero, but that does not diminish its importance.

Example. Solve the population growth problem

$$\frac{dC}{dt} = kC.$$

Treating this as a separable equation, we rearrange to get

$$\frac{dC}{C} = kdt$$

Integrate both sides to get

$$\ln C = kt + K$$

Using the initial condition $C(0)$, we have

$$\begin{aligned} \ln C(0) &= k \cdot 0 + K \\ \ln C &= kt + \ln C(0) \\ C &= C(0)e^{kt} \end{aligned}$$

Example. Solve the equation $\frac{dy}{dx} = (1-2x)y^2$.
We use separation of variables:

$$\begin{aligned} \frac{dy}{y^2} &= (1-2x)dx \\ \int y^{-2}dy &= \int (1-2x)dx \\ -y^{-1} &= x - x^2 + c \\ y &= \frac{1}{x^2 - x - c} \end{aligned}$$

Exercise. A more sophisticated model for population growth was

$$\frac{dC}{dt} = kC\left(1 - \frac{C}{C_{max}}\right)$$

Rearrange the equation to get

$$\frac{dC}{C - C^2/C_{max}} = kdt$$

Hint: use partial fractions.

1.3 Linear equations

1.3.1 Definition

If

$$\frac{dy}{dt} = f(y,t)$$

is linear in y (y appears by itself and not in a function) then the first order equation is said to be linear.
We can't have y^2 or $\sqrt{y+1}$ or anything fancy.
However, we don't care about t, so

$$y' = t^3 y + \sqrt{t}$$

is linear (in y), whereas

$$y' = \frac{t}{y}$$

is not.

1.3.2 Form

A linear first order equation can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

1.3.3 Integrating factors

We can use an integrating factor to make the equation integrable. Consider the function

$$F(t) = \exp\left(\int p(t)dt\right)$$

Note that

$$F'(t) = F(t)p(t)$$

We multiply both sides of the linear equation by $F(t)$.

$$F(t)\left(\frac{dy}{dt} + p(t)y\right) = F(t)q(t)$$

Consider now the expression

$$\frac{d}{dt}(F(t)y(t)) = F(t)\frac{dy}{dt} + \frac{dF}{dt}y(t) \qquad \text{product rule}$$

and so

$$\frac{d}{dy}(F(t)y(t)) = F(t)\frac{dy}{dt} + F(t)p(t)y(t)$$

Thus, the equation we want to solve has become

$$\frac{d}{dt}(F(t)y(t)) = F(t)q(t)$$

which can be integrated to give

$$\int d(F(t)y(t)) = \int F(t)q(t)dt$$

We can now solve to get

$$y(t) = \frac{1}{F(t)}\left(\int F(t)q(t)dt + K\right)$$

where K is a constant of integration.
The function $F(t)$ is called an integrating factor. This uses the product rule for derivatives, but in reverse. Again, the constant of integration is vital.

Example. Solve

$$y' + \frac{1}{1+x}y = 1 + x, \; y(0) = 0.$$

The integrating factor is

$$F(x) = \exp\left(\int \frac{dx}{1+x}\right) = 1 + x$$

Rewrite the equation as

$$(1+x)y' + y = (1+x)^2$$

or

$$\frac{d}{dx}((1+x)y) = (1+x)^2.$$

(See how the product rule "collapses" the left-hand side?)
Integrating both sides, we get

$$((1+x)y) = \frac{1}{3}(1+x)^3 + K$$

where K is an arbitrary constant. Now, solve for y to get

$$y(x) = \frac{1}{3}(1+x)^2 + \frac{K}{1+x}$$

Use the condition $y(0) = 0$ to get

$$K = -\frac{1}{3}$$

Thus the solution is

$$y(x) = \frac{1}{3}(1+x)^2 - \frac{1}{3}\cdot\frac{1}{1+x}$$

1.4 Exact solutions

We can generalise the idea of an integrating factor. What was really going on was that we used the integrating factor to put the equation into an exact form.

From calculus, if a function $u(x, y)$ has continuous partial derivatives, then its total or exact differential is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

In particular, if $u(x, y) = c$ (a constant), then $du = 0$.

Example. Suppose $u = x + x^2 y^3 = c$. Find the differential equation for $\frac{dy}{dx}$.

We have

$$du = (1 + 2xy^3)dx + 3x^2y^2dy = 0.$$

Rearranging,

$$\begin{aligned}(1 + 2xy^3) &= -3x^2y^2\frac{dy}{dx} \\ \frac{dy}{dx} &= -\frac{1 + 2xy^3}{3x^2y^2}.\end{aligned}$$

What we've done here is "solved" a differential equation, only in reverse. This gives us a powerful solution method.

A first-order differential equation of the form

$$\boxed{M(x, y)dx + N(x, y)dy = 0} \tag{1}$$

is called exact if its left-hand side is the total or exact differential

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

of some function $f(x, y)$. Then the differential equation (1) can be written

$$du = 0.$$

By integration, we obtain the general solution of (1) in the form

$$\boxed{u(x, y) = c.} \tag{2}$$

Comparing (1) and (2), we see that (1) is exact if there is some function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M \qquad \text{and} \qquad \frac{\partial u}{\partial y} = N.$$

Suppose M and N are defined and have continuous first partial derivatives in a region of the xy-plane whose boundary is a closed curve having no self-intersections. Then

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial^2 u}{\partial y \partial x} \\ \frac{\partial N}{\partial x} &= \frac{\partial^2 u}{\partial x \partial y}.\end{aligned}$$

By continuity, the two second derivatives are equal. Thus

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

This condition is both necessary and sufficient for $M dx + N dy$ to be an exact differential.

Example. Show that

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy \tag{3}$$

is exact.

We have

$$M = x^3 + 3xy^2 \qquad N = 3x^2y + y^3$$

$$\frac{\partial M}{\partial y} = 6xy \qquad \frac{\partial N}{\partial x} = 6xy$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

If (1) is exact, the function $u(x, y)$ can be found either by guessing or by integrating twice, once with respect to each variable.

Integrating with respect to x, we have

$$u = \int M dx + k(y)$$

where y is regarded as a constant in this integration, so $k(y)$ plays the role of the arbitary constant.

Integrating with respect to y, we have

$$u = \int N dy + j(x)$$

where x is regarded as a constant in this integration, so $j(x)$ plays the role of the arbitary constant.

To determine $j(x)$, we derive $\frac{\partial u}{\partial x}$ to get $\frac{dj}{dx}$ and integrate.

Example. Find the solution to (3). Check that this solution solves the differential equation.

We have

$$\begin{aligned} u &= \int M dx + k(y) \\ &= \int (x^3 + 3xy^2)dx + k(y) \\ &= \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + k(y). \end{aligned}$$

Differentiating this, we have

$$\frac{\partial u}{\partial y} = 3x^2y + \frac{dk}{dy} = N = 3x^2y + y^3.$$

Hence $\frac{dk}{dy} = y^3$. Thus $k = \frac{1}{4}y^4 + \bar{c}$. Thus

$$u(x, y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 = c.$$

To check, we can differentiate u implicitly with respect to x:

$$\begin{aligned} \frac{du}{dx} &= 0 \\ x^3 + 3xy^2 + 3x^2yy' + y^3y' &= 0 \\ x^3 + 3xy^2 + (3x^2y + y^3)\frac{dy}{dx} &= 0. \end{aligned}$$

Example. Solve

$$\cos x \sinh y \frac{dy}{dx} - \sin x \cosh y = 0, \qquad y(0) = 0$$

Check that the solution satisfies the differential equation.

We have

$$M = -\sin x \cosh y \qquad N = \cos x \sinh y$$
$$\frac{\partial M}{\partial y} = -\sin x \sinh y \qquad \frac{\partial N}{\partial x} = -\sin x \sinh y$$

Thus the equation is exact.

To solve, we have

$$\begin{aligned} u &= -\int \sin x \cosh y dx + k(y) \\ &= \cos x \cosh y + k(y) \\ \frac{\partial u}{\partial y} &= \cos x \sinh y + \frac{dk}{dy} = N = \cos x \sinh y. \end{aligned}$$

Thus $\frac{dk}{dy} = 0$ so $k = \bar{c}$.

The general solution is then

$$u(x, y) = \cos x \cosh y = c.$$

The initial condition $y(0) = 0$ gives

$$\cos 0 \cosh 0 = 1 = c.$$

Thus the solution is $\cos x \cosh y = 1$.

Checking, we see that

$$\begin{aligned} (\cos x \cosh y)' &= -\sin x \cosh y + \cos x(\sinh y)y' = 0 \\ \cos 0 \cosh 0 &= 1 \end{aligned}$$

(Don't forget to check the initial condition!)

What happens if the equation isn't exact? In this case, we *cannot* use this technique.

Example. Consider

$$y - xy' = 0$$

Show that the equation is not exact and that the integrating method fails.

We have

$$M = y \qquad N = -x$$
$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1.$$

Thus the equation is not exact, since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Trying the integral method, we have

$$\begin{aligned} u &= \int M dx + k(y) = xy + k(y) \\ \frac{\partial u}{\partial y} &= x + k'(y). \end{aligned}$$

This should equal $N = -x$. But this is impossible, since $k(y)$ can only depend on y.

Exercise. Solve the last equation using a different method.

1.5 The Bernoulli Equation

Some nonlinear differential equations can be reduced to linear forms. The most famous of these is the Bernoulli equation:

$$\boxed{y' + p(x)y = g(x)y^a} \qquad (a \text{ any real number}). \tag{4}$$

If $a = 0$ or $a = 1$, the equation is linear. Otherwise it is nonlinear. In that case, set

$$u(x) = [y(x)]^{1-a}.$$

Differentiating and substituting into (4), we have

$$\begin{aligned} u' = (1-a)y^{-a}y' &= (1-a)y^{-a}(gy^a - py) \\ &= (1-a)(g - py^{1-a}). \end{aligned}$$

Since $u = y^{1-a}$, we thus have a linear equation

$$u' + (1-a)pu = (1-a)g.$$

Example. Solve

$$y' - Ay = -By^2 \qquad (A,B \text{ positive constants}).$$

Here $a = 2$, so $u = y^{-1}$ and we thus have

$$\begin{aligned} u' = -y^{-2}y' &= -y^{-2}(-By^2 + Ay) \\ &= B - Ay^{-1} \\ &= B - Au \\ u' + Au &= B \end{aligned}$$

This is a linear equation, so we can use an integrating factor e^{Ax}:

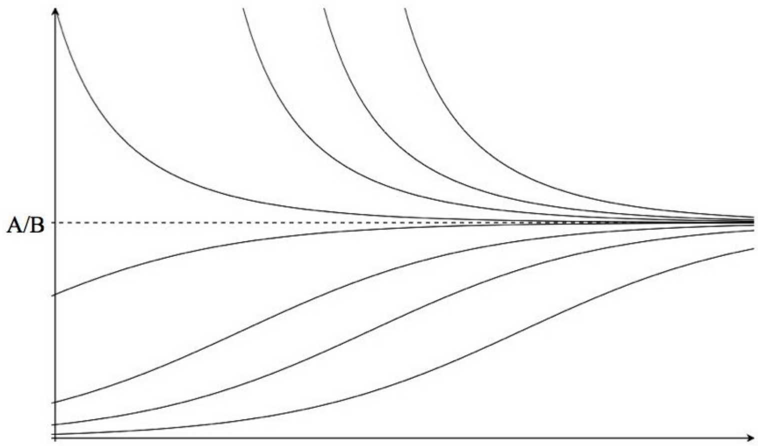
$$\begin{aligned} e^{Ax}u' + Ae^{Ax}u &= Be^{Ax} \\ \frac{d}{dx}(e^{Ax}u) &= Be^{Ax} \\ \int \frac{d}{dx}(e^{Ax}u) &= \int Be^{Ax} \\ e^{Ax}u &= \frac{B}{A}e^{Ax} + c \\ u &= \frac{B}{A} + ce^{-Ax}. \end{aligned}$$

We thus have

$$y = \frac{1}{u} = \frac{1}{\frac{B}{A} + ce^{-Ax}}. \tag{5}$$

Another (trivial) solution is $y = 0$. The solution (5) is called the logistic law of population growth, where x represents time. For $B = 0$, it gives exponential growth $y = (1/c)e^{Ax}$, which is Malthus's law.

Exponential growth is of course not very realistic, as solutions head to infinity very quickly. Thus, the term $-By^2$ in the original differential equation acts as a "braking term", preventing the population from growing without bound. Indeed, small populations (with $0 < y(0) < A/B$) will increase monotonically to A/B, whereas initial conditions with $y(0) > A/B$ will decrease monotonically to the same limit A/B. The value A/B is called the carrying capacity of a population.


A/B

2 Second order linear differential equations

2.1 Second order equations

2.1.1 Form

The general form of a second order linear equation is

$$a(t)y'' + b(t)y' + c(t)y = f(t)$$

Once again, $a(t),\ b(t),\ c(t)$ and $f(t)$ can be nonlinear in t, it's only y we care about. For example,

$$t^3y'' - \cos(t)y' + \ln(t)y = \sqrt{6 + \sin(t)}$$

is a linear second order differential equation.

2.1.2 Homogeneous equation

The equation is homogeneous if $f(t) = 0$, that is, if it is of the form

$$a(t)y'' + b(t)y' + c(t)y = 0$$

For example, $t^3y'' - \cos(t)y' + \ln(t)y = 0$ is a homogeneous second order linear differential equation. Note that cy is also a solution for any c.

2.1.3 Initial and boundary conditions

In order to get a solution which contains no arbitrary constants, we must have two initial conditions or two boundary conditions. For example, we may be given the information

$$y(0) = 0,\ y'(0) = 1$$

or the boundary conditions

$$y(0) = 1.0,\ y(1) = 0.0$$

First-order equations require one condition. Second order equations require two conditions. Initial conditions don't have to be at 0, they just have to be at the same point. For example, $y(3) = 0,\ y'(3) = 1$ are also initial conditions.

2.1.4 Particular solutions

A particular solution y_P to some nonhomogeneous equation is any solution — not necessarily the most general one — to the nonhomogeneous solution.

Example. Find a particular solution to

$$y'' + x^3y' + 2y = 1$$

$$y_P = \frac{1}{2}$$

is a particular solution, but it is certainly not the full solution.

2.2 Complete solutions to the nonhomogeneous equation

The general solution to the nonhomogeneous equation can be written as

$$y = y_h + y_P$$

where y_h is the general solution to the corresponding homogeneous equation and y_P is any particular solution to the nonhomogeneous equation.

The procedure for solving the nonhomogeneous equation is as follows:
1) Find the general solution to the homogeneous equation.
2) Find a particular solution to the original equation.
3) Add the two expressions together.
4) Apply initial or boundary conditions to evaluate the arbitrary constants in y_h.

Example. Solve

$$y' + y = 1 + t, \qquad y(0) = 3$$

First we look at the homogeneous equation:

$$\begin{aligned} y' + y &= 0 \\ y' &= -y \\ y &= ce^{-t} \end{aligned}$$

Next we try and guess a particular solution. In this case, $y = t$ will work, since $y' + y = 1 + t$. Formally, we have

$$\begin{aligned} y_h &= ce^{-t} \\ y_p &= t \\ \therefore\ y &= ce^{-t} + t \\ y(0) &= 3 \to c = 3 \end{aligned}$$

Thus the solution is $y(t) = 3e^{-t} + t$.

2.3 Homogeneous equations

Exercise. Suppose that $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous equation. Show that the linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is also a solution for constants c_1 and c_2.

2.4 Euler-Cauchy equation

Consider the Euler-Cauchy equation

$$t^2 y'' + \alpha t y' + \beta y = 0$$

For now let t be greater than zero.

2.4.1 Guess for Euler-Cauchy equation

Note that if $y = t^r$ then $t^2 y''$ and ty' are multiples of y. So, let's try this as an answer.

$$\begin{aligned} y &= t^r \\ y' &= rt^{r-1} \\ y'' &= r(r-1)t^{r-2} \end{aligned}$$

Substitute back into the original equation to get

$$r(r-1)t^r + \alpha r t^r + \beta t^r = 0$$

2.4.2 Indicial equation

We get

$$F(r)t^r = 0$$

where

$$F(r) = r(r-1) + \alpha r + \beta$$

and so we have the indicial equation

$$F(r) = r(r-1) + \alpha r + \beta = r^2 + (\alpha - 1)r + \beta = 0$$

2.4.3 Roots of the indicial equation

The indicial equation has two roots

$$r_1 = -\frac{1}{2}\left[(\alpha-1)+\sqrt{(\alpha-1)^2-4\beta}\right]$$
$$r_2 = -\frac{1}{2}\left[(\alpha-1)-\sqrt{(\alpha-1)^2-4\beta}\right]$$

2.4.4 Two real roots

$$y(t) = At^{r_1} + Bt^{r_2}$$

Example. Solve

$$t^2y'' + 4ty' + 2y = 0.$$

Letting $y = t^r$, we have

$$\begin{aligned} r(r-1)+4r+2 &= 0 \\ r^2+3r+2 &= 0 \\ (r+1)(r+2) &= 0 \\ r &= -1,-2. \end{aligned}$$

Thus

$$y(t) = At^{-1} + Bt^{-2}$$

2.4.5 Two equal roots

Suppose that the indicial equation has two equal roots. Then the solution is of the form

$$y(t) = At^r + B\log(t)t^r$$

We can see this if we let $t = e^z$. Then

$$\begin{aligned} dt = e^z dz \qquad \text{so} \qquad \frac{dy}{dt} &= \frac{1}{e^z}\frac{dy}{dz} \\ \text{and} \qquad \frac{d^2y}{dt^2} &= \frac{1}{e^{2z}}\frac{d^2y}{dz^2} \\ \therefore t^2y'' + \alpha ty' + \beta y = 0 &\text{ becomes} \\ e^{2z}\frac{1}{e^{2z}}\frac{d^2y}{dz^2} + \alpha e^z\frac{1}{e^z}\frac{dy}{dz} + \beta y &= 0 \\ \frac{d^2y}{dz^2} + \alpha\frac{dy}{dz} + \beta y &= 0 \text{ - constant coefficients} \end{aligned}$$

For the case of two equal roots with constant coefficients, solutions are of the form

$$y = Ae^{rz} + Bze^{rz}$$

where $r^2 + \alpha r + \beta = 0$ and $r = \frac{-\alpha\pm\sqrt{\alpha^2-4\beta}}{2} = -\frac{\alpha}{2}$ since the roots are repeated.

Check:

$$\begin{aligned} y' &= Are^{rz} + Be^{rz} + Brze^{rz} \\ y'' &= Ar^2e^{rz} + Bre^{rz} + Bre^{rz} + Br^2ze^{rz} \\ &= Ar^2e^{rz} + 2Bre^{rz} + Br^2ze^{rz} \\ y'' + \alpha y' + \beta y &= Ar^2e^{rz} + 2Bre^{rz} + Br^2ze^{rz} + \alpha(Are^{rz} + Be^{rz} \\ &\quad + Brze^{rz}) + \beta(Ae^{rz} + Bze^{rz}) \\ &= Ae^{rz}(r^2+\alpha r+\beta) + Be^{rz}(2r+\alpha) \\ &\quad + Bze^{rz}(r^2+\alpha r+\beta) = 0 \end{aligned}$$

Thus, since $t = e^z$,

$$y = At^r + B\ln(t)t^r$$

Example.

$$t^2y'' - 5ty' + 9y = 0$$

Show that the indicial equation has equal real roots $r = 3$. What is the general solution?

$$\begin{aligned} y &= t^r \\ r(r-1)t^r - 5rt^r + 9t^r &= 0 \\ r^2 - r - 5r + 9 &= 0 \\ r^2 - 6r + 9 &= 0 \\ (r-3)^2 &= 0 \\ y &= At^3 + B\log(t) - t^3 \end{aligned}$$

2.4.6 Complex roots

Suppose that the solutions to the indicial equation are

$$r = \lambda \pm i\mu$$

Then the solutions are

$$y(t) = t^\lambda[A\cos(\mu\log(t)) + B\sin(\mu\log(t))]$$

We can see this from letting $t = e^z$ again. Then

$$\begin{aligned} y &= c_1e^{\lambda z}e^{\mu iz} + c_2e^{\lambda z}e^{-\mu iz} \\ &= e^{\lambda z}(c_1\cos(\mu z) + ic_1\sin(\mu z) + c_2\cos(\mu z) - c_2 i\sin(\mu z)) \\ &= e^{\lambda z}(A\cos(\mu z) + B\sin(\mu z)) \\ &= t^\lambda(A\cos(\mu\log(t)) + B\sin(\mu\log(t))) \end{aligned}$$

Example. Solve $t^2y'' - 5ty' + 25y = 0$.

$$\begin{aligned} 0 &= t^2y'' - 5ty' + 25y \\ 0 &= r(r-1)t^r - 5rt^r + 25t^r \\ 0 &= r^2 - r - 5r + 25 \\ 0 &= r^2 - 6r + 25 \\ r &= \frac{6 \pm \sqrt{36 - 100}}{2} \\ &= \frac{6 \pm 8i}{2} \\ &= 3 \pm 4i \\ y &= e^{3z}(A\cos(4z) + B\sin(4z)) \\ &= t^3(A\cos(4\log(t)) + B\sin(4\log(t))) \end{aligned}$$

2.4.7 The Wronskian

The Wronskian of two functions u_1 and u_2 is defined as

$$W(u_1, u_2) = \det\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} = u_1u_2' - u_1'u_2$$

2.4.8 Linear independence

Two solutions are linearly independent if the Wronskian of the two functions is not zero.

Example. Show that $y(t) = t$ and $y(t) = 1 + t$ are two independent solutions of $y'' = 0$, but that $3 - 2t$ and $6t - 9$ are not.

Clearly, all the functions solve the differential equation.

$$W(t, 1+t) = \det \begin{pmatrix} t & 1+t \\ 1 & 1 \end{pmatrix} = t - 1 - t = -1$$

so t and $1 + t$ are independent.

Conversely,

$$W(3-2t, 6t-9) = \det \begin{pmatrix} 3-2t & 6t-9 \\ -2 & 6 \end{pmatrix} = 18 - 12t + 12t - 18 = 0$$

so $3 - 2t$ and $6t - 9$ are dependent. (In fact, one is simply a multiple of the other.)

2.4.9 General solution to the homogeneous equation

If $y_1(t)$ and $y_2(t)$ are linearly independent solutions of

$$a(t)y'' + b(t)y' + c(t)y = 0$$

then the general solution is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

where c_1 and c_2 are arbitrary constants. The set $\{y_1, y_2\}$ is called the fundamental set of solutions. Two solutions are linearly independent if and only if one is not a constant multiple of the other.

2.5 Homogeneous equations with constant coefficients

2.5.1 Form

We consider equations of the form

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and a is not zero.

2.5.2 Guessing a solution

We need to find two independent solutions. Let's guess a solution of the form

$$y = e^{rt}$$

Substitute this solution into the equation, we get

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

Now divide by the common factor to get the characteristic equation.

$$ar^2 + br + c = 0$$

Thus $y = e^{rt}$ is a solution to the second order linear homogeneous equation with constant coefficients for values of r which satisfy the characteristic equation.
The roots of the characteristic equation are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can always divide by e^{rt} since it is never zero (even when r is complex).

Example. Solve $y' + y = 0$ (not a second order equation obviously).

We have $r + 1 = 0 \to r = -1$. So a solution is $y = e^{-t}$ and the general solution consists of all scalar multiples of this: $y = ce^{-t}$.

2.5.3 First possibility

If the discriminant is positive

$$b^2 - 4ac > 0$$

then we have two real and distinct roots r_1 and r_2 and two linearly independent solutions

$$y_1 = e^{r_1 t}$$
$$y_2 = e^{r_2 t}$$

and so the general solution to the homogeneous equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Example. Solve $y'' - 5y' + 6y = 0$

The characteristic equation is $r^2 - 5r + 6 = 0$, so $r = 2, 3$. Thus the general solution is $y = c_1 e^{2t} + c_2 e^{3t}$.

2.5.4 Second possibility

If the discriminant is zero

$$b^2 - 4ac = 0$$

then the roots are equal: $r_1 = r_2 = r$. One solution is

$$y_1 = e^{rt}$$

and we already saw that another (clearly linearly independent) solution is

$$y_2 = te^{rt}.$$

Thus the general solution is

$$y = c_1 e^{rt} + c_2 te^{rt}.$$

We have

$$\begin{aligned}
y &= te^{rt} \\
y' &= e^{rt} + rte^{rt} \\
y'' &= 2re^{rt} + r^2 te^{rt} \\
ay'' + by' + cy &= 2are^{rt} + ar^2 te^{rt} + be^{rt} + brte^{rt} + cte^{rt} \\
&= (2ar + b)e^{rt} + (ar^2 + br + c)te^{rt} \\
&= (0)e^{rt} + (0)te^{rt} = 0 \text{ because in this case } r = -\frac{b}{2a}
\end{aligned}$$

i.e. $y = te^{rt}$ is a solution, but <u>only</u> in the case of repeated roots.

$$\begin{aligned}
W(e^{rt}, te^{rt}) &= \det \begin{pmatrix} e^{rt} & te^{rt} \\ re^{rt} & e^{rt} + rte^{rt} \end{pmatrix} \\
&= e^{2rt} + rte^{2rt} - rte^{2rt} \\
&= e^{2rt} \neq 0
\end{aligned}$$

Therefore, e^{rt} and te^{rt} are independent.

Example. Solve $y'' - 6y' + 9y = 0$.

We have

$$\begin{aligned}
r^2 - 6r + 9 &= 0 \\
(r - 3)^2 &= 0 \\
r &= 3, 3
\end{aligned}$$

Thus

$$y = c_1 e^{3t} + c_2 te^{3t}$$

2.5.5 Third possibility

If the discriminant is negative

$$b^2 - 4ac < 0$$

then the two roots are complex conjugates

$$r_1 = \alpha + i\beta \qquad\qquad r_2 = \alpha - i\beta$$

and, from Euler's identity,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we get the general solution

$$y = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)]$$

In polar coordinates, we have

$$\begin{aligned}
y &= c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t} \\
&= c_1 e^{\alpha t} e^{i\beta t} + c_2 e^{\alpha t} e^{-i\beta t} \\
&= e^{\alpha t}(c_1\cos(\beta t) + ic_1\sin(\beta t) + c_2\cos(\beta t) - ic_2\sin(\beta t)) \\
&= e^{\alpha t}[(c_1 + c_2)\cos(\beta t) + i(c_1 - c_2)\sin(\beta t)]
\end{aligned}$$

Since c_1 and c_2 are arbitrary, we could choose them to be complex conjugates, so that $c_1 + c_2$ is real and $c_1 - c_2$ is purely imaginary. Thus, even though the roots of the equation are complex, it is possible to get real solutions, of the form

$$y = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)]$$

Example. Solve $y'' + 6y' + 13y = 0$.
We have

$$\begin{aligned}
r^2 + 6r + 13 &= 0 \\
r &= \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i \\
y &= e^{-3t}(A\cos(2t) + B\sin(2t)).
\end{aligned}$$

Example. Show that the general solution for $y'' + \alpha^2 y = 0$ for α positive is $y = A\cos(\alpha t) + B\sin(\alpha t)$.
The characteristic equation is

$$\begin{aligned}
r^2 + \alpha^2 &= 0 \\
r^2 &= -\alpha^2 \\
r &= \pm i\alpha \\
y &= c_1 e^{i\alpha t} + c_2 e^{-i\alpha t} \\
&= c_1\cos(\alpha t) + ic_1\sin(\alpha t) + c_2\cos(\alpha t) - c_2 i\sin(\alpha t) \\
&= A\cos(\alpha t) + B\sin(\alpha t) \\
A &= c_1 + c_2 \\
B &= (c_1 - c_2)i
\end{aligned}$$

Example. Solve the initial value problem $y'' + y' - 2y = 0$ with initial conditions $y(0) = 1,\ y'(0) = 1$.

We have

$$\begin{aligned} m^2 + m - 2 &= 0 \\ (m+2)(m-1) &= 0 \\ m &= -2,\ 1 \\ y &= c_1 e^{-2t} + c_2 e^t \\ y(0) &= c_1 + c_2 = 1 \\ y' &= -2c_1 e^{-2t} + c_2 e^t \\ y'(0) &= -2c_1 + c_2 = 1 \\ -2c_1 + (1 - c_1) &= 1 \\ c_1 &= 0 \\ c_2 &= 1 \\ y &= e^t \end{aligned}$$

Exercise. Find the solution to

$$y'' + y = 0.$$

2.6 Nonhomogeneous equations with constant coefficients

2.6.1 Form

$$ay'' + by' + cy = f(t)$$

where $a,\ b$ and c are constants and $f(t)$ is a given function of t.

2.6.2 Solutions

Solutions are of the form

$$y(t) = y_h(t) + y_p(t)$$

where y_h is the general solution of the homogeneous equation obtained by setting $f(t)$ to zero and y_p is some particular solution.
We already discussed how to find y_h. How do we find some y_p? We discuss two methods: the method of undetermined coefficients which is easy if it works, and the method of variation of parameters which is more cumbersome but always works, at least formally.

2.6.3 Method of undetermined coefficients

The idea is to guess the form of the solution with some unknown coefficients which then have to be found.
Example. Find a particular solution to

$$y'' + y = t^2.$$

In this case, $f(t)$ is a polynomial in t of degree 2, so we guess that a particular solution has such a form. Let

$$y = At^2 + Bt + C$$

and then we get

$$\begin{aligned} y' &= 2At + B \\ y'' &= 2A \end{aligned}$$

Subtituting into the equation gives

$$2A + At^2 + Bt + C = t^2$$

Equating coefficients of powers of t on both sides gives

$$\begin{aligned} 2A + C &= 0 \\ B &= 0 \\ A &= 1 \end{aligned}$$

which has solutions $A = 1$, $B = 0$, and $C = -2$.
Then a particular solution is

$$y_p = t^2 - 2.$$

Exercise. Show that the solution to

$$y'' + y = t^2$$

is

$$y = C_1 \cos t + C_2 \sin t + t^2 - 2$$

Example. Solve

$$y'' + y' + y = \cos t.$$

First we need a particular solution.
We guess a particular solution of the form

$$y_P = A \cos t + B \sin t$$

because derivatives of sin and cos are sin and cos, so combinations lead to $\cos t$.

$$\begin{aligned} y_p &= A \cos t + B \sin t \\ y_p' &= -A \sin t + B \cos t \\ y_p'' &= -A \cos t - B \sin t \\ y_p'' + y_p' + y_p &= -A \cos t - B \sin t - A \sin t + B \cos t + A \cos t + B \sin t \\ &= (-A + B + A) \cos t + (-B - A + B) \sin t \\ &= B \cos t - A \sin t = \cos t \\ \therefore B &= 1, \ A = 0 \\ y_p &= (0) \cos t + (1) \sin t \\ y_p &= \sin t \end{aligned}$$

We also need the homogeneous solution.
The characteristic equation is

$$\begin{aligned} 0 &= r^2 + r + 1 \\ r &= \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2} \\ y_h &= e^{-t/2} \left(c_1 \cos \left(\frac{\sqrt{3}}{2} t \right) + c_2 \sin \left(\frac{\sqrt{3}}{2} t \right) \right) \end{aligned}$$

Hence the solution is

$$y = e^{-t/2} \left(c_1 \cos \left(\frac{\sqrt{3}}{2} t \right) + c_2 \sin \left(\frac{\sqrt{3}}{2} t \right) \right) + \sin t$$

Example. Solve

$$y'' + 4y = 7e^{-3x} \cos 5x$$

First let's find the homogeneous solution.
The characteristic equation is $m^2 + 4 = 0$ so $m = \pm 2i$. Thus

$$\begin{aligned} y_h &= c_1 e^{2ix} + c_2 e^{-2ix} \\ &= A\cos 2x + B\sin 2x. \end{aligned}$$

Next we look for a particular solution.
We'll guess a particular solution of the form

$$y_p = e^{-3x}(P\cos 5x + Q\sin 5x).$$

Then we have

$$\begin{aligned} y_p' &= -3e^{-3x}(P\cos 5x + Q\sin 5x) + e^{-3x}(-5P\sin 5x + 5Q\cos 5x) \\ y_p'' &= 9e^{-3x}(P\cos 5x + Q\sin 5x) - 3e^{-3x}(-5P\sin 5x + 5Q\cos 5x) \\ &\quad - 3e^{-3x}(-5P\sin 5x + 5Q\cos 5x) + e^{-3x}(-25P\cos 5x - 25Q\sin 5x) \\ &= e^{-3x}(-16P\cos 5x - 16Q\sin 5x) + e^{-3x}(30P\sin 5x - 30Q\cos 5x) \end{aligned}$$

Substituting, we have

$$\begin{aligned} y_p'' + 4y_p &= e^{-3x}(-16P\cos 5x - 16Q\sin 5x) + e^{-3x}(30P\sin 5x - 30Q\cos 5x) + 4[e^{-3x}(P\cos 5x + Q\sin 5x)] \\ &= e^{-3x}(-12P\cos 5x - 12Q\sin 5x) + e^{-3x}(30P\sin 5x - 30Q\cos 5x) \\ &= e^{-3x}[(-12P - 30Q)\cos 5x + (30P - 12Q)\sin 5x] = 7e^{-3x}\cos 5x. \end{aligned}$$

It follows that

$$\begin{aligned} -12P - 30Q &= 7 \\ 30P - 12Q &= 0 \end{aligned}$$

Thus

$$\begin{aligned} Q &= \frac{5}{2}P \\ -12P - 30\left(\frac{5}{2}P\right) &= 7 \\ P &= -\frac{7}{87} \\ Q &= -\frac{35}{174} \end{aligned}$$

Thus the solution is

$$\begin{aligned} y &= y_h + y_p \\ &= A\cos 2x + B\sin 2x - \frac{7}{87}e^{-3x}\cos 5x - \frac{35}{174}e^{-3x}\sin 5x \end{aligned}$$

The method of undetermined parameters can thus be summarised as follows:

Term in $r(x)$	Choice for y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, \dots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ and/or $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ and/or $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

2.6.4 A complication

The situation becomes more complicated if $f(t)$ contains one of the fundamental solutions of the homogeneous equation. For example, consider

$$y'' + y = \cos t$$

Since $f(t)$ contains a cos function it would be reasonable to guess

$$y_p = A\cos t + B\sin t$$

However, substituting this into the original equation gives

$$0 = \cos t$$

We extend the procedure and use a guess of the form

$$y_p = At\cos t + Bt\sin t$$

Example. Complete this problem.
First we find the particular solution.

$$\begin{aligned}
y_p' &= A\cos t - At\sin t + B\sin t + Bt\cos t \\
&= (A + Bt)\cos t + (B - At)\sin t \\
y_p'' &= B\cos t - (A + Bt)\sin t - A\sin t + (B - At)\cos t \\
&= (2B - At)\cos t - (2A + Bt)\sin t \\
y_p'' + y_p &= (2B - At)\cos t - (2A + Bt)\sin t + At\cos t + Bt\sin t \\
&= 2B\cos t - 2A\sin t = \cos t
\end{aligned}$$

Thus $B = \frac{1}{2}$ and $A = 0$, so the particular solution is

$$y_p = \frac{1}{2}t\sin t$$

Next, we find the homogeneous solution.
The characteristic equation for the homogeneous solution is

$$\begin{aligned}
r^2 + 1 &= 0 \\
r &= \pm i
\end{aligned}$$

so the homogeneous solution is

$$y_h = c_1\cos t + c_2\sin t.$$

Thus, the solution is

$$y = c_1\cos t + c_2\sin t + \frac{1}{2}t\sin t.$$

Example. Solve $y'' + 4y = 3\sin(2t)$.

$$\begin{aligned}
y'' + 4y &= 3\sin(2t) \\
m^2 + 4 &= 0 \\
m &= \pm 2i \\
y_h &= c_1\cos(2t) + c_2\sin(2t)
\end{aligned}$$

If we try the obvious particular solution, it won't work:

$$\begin{aligned}
y_p &= A\cos(2t) + B\sin(2t) \\
y_p' &= -2A\sin(2t) + 2B\cos(2t) \\
y_p'' &= -4A\cos(2t) - 4B\sin(2t) \\
y_p'' + 4y_p &= -4A\cos(2t) - 4B\sin(2t) + 4A\cos(2t) + 4B\sin(2t) \\
\to 0 &= 3\sin(2t)
\end{aligned}$$

This doesn't work, because the particular solution we tried had the same form as the homogeneous solution.

Thus, let's try the same thing, only with a factor of t:

$$\begin{aligned}
y_p &= At\cos(2t) + Bt\sin(2t) \\
y_p' &= A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t) \\
y_p'' &= -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) + 2B\cos(2t) + 2B\cos(2t) - 4Bt\sin(2t) \\
y_p'' + 4y_p &= -4A\sin(2t) - 4At\cos(2t) + 4B\cos(2t) - 4Bt\sin(2t) + 4At\cos(2t) + 4Bt\sin(2t) \\
&= -4A\sin(2t) + 4B\cos(2t) \\
&= 3\sin(2t) \\
A &= -\frac{3}{4}, \ B = 0 \\
y_p &= -\frac{3}{4}t\cos(2t) \\
y &= c_1\cos(2t) + c_2\sin(2t) - \frac{3}{4}t\cos(2t)
\end{aligned}$$

Exercise. Consider

$$y'' - y = \cos t + e^{-t}$$

Try to find a particular solution of the form

$$y_p = A\cos t + B\sin t + Ce^{-t} + Dte^{-t}$$

Why do we need the last term?

We can guess particular solutions when we know what sort of functions differentiate into the nonhomogeneous part (often differentiate multiple times). Specifically polynomials, exponentials and trigonometric functions. What happens when we don't know this?

2.6.5 Example of variation of parameters

We do an example before discussing the method in general. Consider

$$y'' + y = \sin^2 t$$

The solution to the homogeneous equation is

$$y_h = C_1\cos t + C_2\sin t$$

We guess that the particular solution has the form

$$y_p = v_1(t)\cos t + v_2(t)\sin t$$

where v_1 and v_2 are unknown functions of t. Differentiating gives

$$y_p' = -v_1\sin t + v_2\cos t + v_1'\cos t + v_2'\sin t$$

Now let the terms involving v_1' and v_2' be equal to zero

$$v_1'\cos t + v_2'\sin t = 0 \qquad (*)$$

Take the second derivative

$$y_p'' = -v_1 \cos t - v_2 \sin t - v_1' \sin t + v_2' \cos t$$

and substitute into the original equation to get

$$-v_1' \sin t + v_2' \cos t = \sin^2 t \qquad (**)$$

The equations $(*)$ and $(**)$ can be solved to give

$$v_1' = -\sin^3 t \qquad v_2' = \cos t \sin^2 t$$

Thus,

$$v_1 = -\int \sin^3 t \; dt \qquad \text{and} \qquad v_2 = \int \cos t \sin^2 t dt$$

We can use substitution for each integral.

$$\begin{aligned} v_1 &= -\int (1-\cos^2 t)\sin t dt \\ &= -\int (1-u^2)\sin t \frac{du}{-\sin t} \\ &= u - \frac{u^3}{3} \\ &= -\frac{1}{3}\cos^3 t + \cos t \end{aligned} \qquad \begin{aligned} u &= \cos t \\ \frac{du}{dt} &= -\sin t \\ dt &= \frac{du}{-\sin t} \end{aligned}$$

$$\begin{aligned} v_2 &= \int \cos t \sin^2 t dt \\ &= \int \cos t u^2 \frac{du}{\cos t} \\ &= \frac{u^3}{3} \\ &= \frac{\sin^3 t}{3} \end{aligned} \qquad \begin{aligned} u &= \sin t \\ \frac{du}{dt} &= \cos t \\ dt &= \frac{du}{\cos t} \end{aligned}$$

Thus, the particular solution is

$$y_p = -\frac{1}{3}\cos^4 t + \cos^2 t + \frac{1}{3}\sin^4 t$$

To guess y_p, we "vary the parameters" of the homogeneous part. There is one equation and two unkown functions, so there will be lots of functions that work.

2.6.6 Method of the variation of parameters

Consider the second order linear equation with variable coefficients:

$$y'' + p(t)y' + q(t)y = f(t)$$

Let y_1 and y_2 be two fundamental solutions to the homogeneous equation. Then let the particular solution be of the form

$$y_p = v_1 y_1 + v_2 y_2$$

Differentiate once to get

$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$$

Then set the last two terms equal to zero so that we get the first equation for v_1 and v_2

$$v_1' y_1 + v_2' y_2 = 0 \qquad (*)$$

Now differentiate y_p' once more to get

$$y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

Substitute into the equation to be solved and rearrange to get

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = f(t)$$

Since y_1 and y_2 are solutions of the homogeneous equation the bracketed terms vanish and one gets the second equation

$$v_1'y_1' + v_2'y_2' = f(t) \qquad (**)$$

We can solve $(*)$ and $(**)$ to get

$$v_1' = -\frac{y_2 f}{W(y_1, y_2)} \qquad \text{and} \qquad v_2' = \frac{y_1 f}{W(y_1, y_2)}$$

and then integrate to get

$$v_1 = -\int \frac{y_2 f}{W(y_1, y_2)} dt \qquad \text{and} \qquad v_2 = \int \frac{y_1 f}{W(y_1, y_2)} dt$$

and so the particular solution is

$$y_p = -y_1 \int \frac{y_2 f}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 f}{W(y_1, y_2)} dt$$

The general solution is then

$$y = C_1 y_1 + C_2 y_2 + y_p$$

To solve $(*)$ and $(**)$, we have

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= f \end{aligned}$$
$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Now we use Cramer's rule

$$v_1' = \frac{\det \begin{pmatrix} 0 & y_2 \\ f & y_2' \end{pmatrix}}{W(y_1, y_2)} = \frac{-y_2 f}{W(y_1, y_2)}$$
$$v_2' = \frac{\det \begin{pmatrix} y_1 & 0 \\ y_1' & f \end{pmatrix}}{W(y_1, y_2)} = \frac{y_1 f}{W(y_1, y_2)}$$

In this case, we don't need to worry about constants of integration since they are absorbed into c_1 and c_2. Conversely, the whole solution comes out of y_p by having arbitrary constants in the integral. The only reason we don't do this is because then it's not really a "particular" solution.

Example. Solve $y'' + 4y = 3\sin(2t)$ by variation of parameters.

First we find v_1 and v_2.

$$
\begin{aligned}
y_p &= v_1(t)\cos(2t) + v_2(t)\sin(2t)\\
y_p' &= -2v_1(t)\sin(2t) + 2v_2(t)\cos(2t) + v_1'(t)\cos(2t) + v_2'(t)\sin(2t)\\
0 &= v_1'(t)\cos(2t) + v_2'(t)\sin(2t) \qquad (*)\\
y_p'' &= -4v_1(t)\cos(2t) - 4v_2(t)\sin(2t) - 2v_1'(t)\sin(2t) + 2v_2'(t)\cos(2t)\\
y_p'' + 4y_p &= -4v_1(t)\cos(2t) - 4v_2(t)\sin(2t) - 2v_1'(t)\sin(2t) + 2v_2'(t)\cos(2t)\\
&\quad + 4v_1(t)\cos(2t) + 4v_2(t)\sin(2t)\\
&= -2v_1'(t)\sin(2t) + 2v_2'(t)\cos(2t) = 3\sin(2t) \qquad (**)
\end{aligned}
$$

$$
\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} \begin{pmatrix} v_1'(t) \\ v_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 3\sin(2t) \end{pmatrix}
$$

$$
\begin{aligned}
v_1'(t) &= \frac{\det\begin{pmatrix} 0 & \sin(2t) \\ 3\sin(2t) & 2\cos(2t) \end{pmatrix}}{\det\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix}} && \begin{array}{c}\text{(Cramer's rule)}\\ \leftarrow (W(\cos(2t), \sin(2t))\end{array}\\
&= \frac{-3\sin^2(2t)}{2\cos^2(2t) + 2\sin^2(2t)}\\
&= -\frac{3}{2}\sin^2(2t)\\
v_2'(t) &= \frac{\det\begin{pmatrix} \cos(2t) & 0 \\ -2\sin(2t) & 3\sin(2t) \end{pmatrix}}{2}\\
&= \frac{3}{2}\cos(2t)\sin(2t)\\
v_1 &= -\frac{3}{2}\int \sin^2(2t)dt\\
&= -\frac{3}{2}\int \frac{1}{2}(1-\cos(4t))dt && \text{(double angle formula)}\\
&= -\frac{3}{4}\left(t - \frac{\sin(4t)}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
v_2 &= \frac{3}{2}\int \cos(2t)\sin(2t)dt && u = \sin(2t)\\
&= \frac{3}{2}\int \cancel{\cos(2t)}u\frac{du}{2\cancel{\cos(2t)}} && \frac{du}{dt} = 2\cos(2t)\\
&= \frac{3}{4}\frac{u^2}{2} && dt = \frac{du}{2\cos(2t)}\\
&= \frac{3}{8}\sin^2(2t)
\end{aligned}
$$

We thus have the particular solution:

$$
\begin{aligned}
y_p &= -\frac{3}{4}\left(t - \frac{\sin(4t)}{4}\right)\cos(2t) + \frac{3}{8}\sin^3(2t)\\
&= -\frac{3}{4}t\cos(2t) + \frac{3}{16}\left[\sin(4t)\cos(2t) + 2\sin^3(2t)\right]\\
&= -\frac{3}{4}t\cos(2t) + \frac{3}{16}\left[2\sin(2t)\cos^2(2t) + 2\sin^3(2t)\right]\\
&= -\frac{3}{4}t\cos(2t) + \frac{3}{16}\left[2\sin(2t)(1-\sin^2(2t)) + 2\sin^3(2t)\right]\\
&= -\frac{3}{4}t\cos(2t) + \frac{3}{8}\sin(2t)
\end{aligned}
$$

Hence the solution is

$$\begin{aligned} y &= c_1 \cos(2t) + c_2 \sin(2t) - \frac{3}{4} t \cos(2t) + \frac{3}{8} \sin(2t) \\ &= c_1 \cos(2t) + \tilde{c_2} \sin(2t) - \frac{3}{4} t \cos(2t) \end{aligned}$$

2.7 Free oscillations

2.7.1 Model of mass-spring system

A mass-spring system with damping but no external force can be modelled by

$$my'' + cy' + ky = 0$$

where

$$\begin{aligned} m &= \text{mass} \\ c &= \text{damping constant} \\ k &= \text{spring constant} \end{aligned}$$

All of these parameters must be nonnegative.
The initial conditions are $y(0) = \alpha$ and $y'(0) = \beta$.

Example. The case of no friction is modelled by setting $c = 0$. Show that the general solution is

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

where the frequency ω_0 is given by

$$\omega_0^2 = \frac{k}{m}$$

We have

$$\begin{aligned} my'' + ky &= 0 \\ mr^2 + k &= 0 \quad \text{is the characteristic equation} \\ r^2 &= -\frac{k}{m} \\ r &= \pm\sqrt{\frac{k}{m}} i = \pm\omega_0 i \\ y &= C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \end{aligned}$$

Example. Show that the solution can be rewritten in the form

$$y(t) = A \cos(\omega_0 t - \phi)$$

where

$$A = \sqrt{C_1^2 + C_2^2}$$

and

$$\tan\phi = \frac{C_2}{C_1}$$

A is called the amplitude, ω_0 is the circular frequency, and ϕ is called the phase angle.

Work backwards:

$$y(t) = A\cos(\omega_0 t - \phi) = A\left[\cos(\omega_0 t)\cos(\phi) + \sin(\omega_0 t)\sin(\phi)\right]$$
$$A\cos\phi = C_1$$
$$A\sin\phi = C_2$$
$$A^2\cos^2\phi + A^2\sin^2\phi = C_1^2 + C_2^2$$
$$A = \sqrt{C_1^2 + C_2^2}$$
$$\tan\phi = \frac{C_2}{C_1}$$

2.7.2 Simple harmonic motion

The frictionless motion just described is called simple harmonic motion. The period T is the amount of time it takes the mass to complete one full oscillation and is

$$T = \frac{2\pi}{\omega_0}$$

and the frequency f is

$$f = \frac{1}{T} = \frac{\omega_0}{2\pi}$$

Example. Show that damped oscillations (ie $c > 0$) give rise to three distinct types of solution.

The characteristic equation is

$$mr^2 + cr + k = 0$$

For $c > 0$, the characteristic equation has the solutions

$$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Three different kinds of solutions arise:

1. $c^2 - 4km > 0$ gives r_1 and r_2 real and different values so $y(t) = C_1e^{r_1t} + C_2e^{r_2t}$.
2. $c^2 - 4km = 0$ gives real and equal roots $r_1, r_2 = \frac{-c}{2m}$ so $y(t) = C_1e^{-\frac{c}{2m}t} + C_2te^{-\frac{c}{2m}t}$.
3. $c^2 - 4km < 0$ gives two complex conjugate roots so

$$\begin{aligned}
y &= C_1e^{\frac{-c+\sqrt{4mk-c^2}i}{2m}t} + C_2e^{\frac{-c-\sqrt{4mk-c^2}i}{2m}t} \\
&= e^{\frac{-ct}{2m}}\left(C_1e^{\mu it} + C_2e^{-\mu it}\right) \\
&= e^{\frac{-ct}{2m}}\left(C_1\cos(\mu t) + C_1i\sin(\mu t) + C_2\cos(\mu t) - C_2i\sin(\mu t)\right) \\
&= e^{\frac{-ct}{2m}}\left(A\cos(\mu t) + B\sin(\mu t)\right) \\
\text{where } \mu &= \frac{\sqrt{4km - c^2}}{2m}
\end{aligned}$$

In all three cases, the real roots are negative and so $y(t)$ converges to zero.

In the first case, the system is overdamped. The amplitude of the motion goes to zero and there are no oscillations.

In the second case, the system is said to be critically damped.

In the third case, the system is underdamped. Oscillations occur and the amplitude of the oscillations decreases exponentially with t.

2.8 Forced damped vibrations

2.8.1 Model

Consider a mass-spring system to which some driving sinusoidal force is applied. Then the model equation becomes

$$mu'' + cu' + ku = F_0 \cos(\omega t)$$

Example. Assuming $c^2 \neq 4mk$, show that the general solution is of the form

$$u = c_1 e^{r_1 t} + c_2 e^{r_2 t} + R\cos(\omega t - \delta)$$

and find R and δ.

The characteristic equation is

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$u_h = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad \text{where this may be complex exponential}$$

$$u_p = A\cos(\omega t) + B\sin(\omega t)$$

$$u_p' = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$

$$u_p'' = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

$$-m\omega^2 A\cos(\omega t) - m\omega^2 B\sin(\omega t) - c\omega A\sin(\omega t) + c\omega B\cos(\omega t) + kA\cos(\omega t) + kB\sin(\omega t) = F_0\cos(\omega t)$$

$$(-m\omega^2 A + c\omega B + kA)\cos(\omega t) + (-m\omega^2 B - c\omega A + kB)\sin(\omega t) = F_0\cos(\omega t)$$

$$A = \frac{-m^2\omega^2 + k}{c\omega}B \qquad \therefore\ B = \frac{c\omega}{-m\omega^2 + k}A$$

$$F_0 = -m\omega^2 A + \frac{c^2\omega^2}{-m\omega^2 + k}A + kA$$

$$F_0(-m\omega^2 + k) = (m^2\omega^4 - km\omega^2 + c^2\omega^2 - m\omega^2 k + k^2)A$$

$$F_0(-m\omega^2 + k) = ((-m\omega^2 + k)^2 + c^2\omega^2)A$$

$$A = \frac{F_0(-m\omega^2 + k)}{\Delta}$$

$$B = \frac{F_0 c\omega}{\Delta}$$

$$u_p = \frac{F_0(-m\omega^2 + k)}{\Delta}\cos(\omega t) + \frac{F_0 c\omega}{\Delta}\sin(\omega t)$$

We can rewrite this in the form

$$u_p = R\cos(\omega t - \delta)$$

where R and δ are two unknowns.

Check:

$$R\cos(\omega t - \delta) = R\cos(\omega t)\cos(\delta) + R\sin(\omega t)\sin(\delta)$$

$$R\cos(\delta) = \frac{F_0(-m\omega^2 + k)}{\Delta}$$

$$R\sin(\delta) = \frac{F_0 c\omega}{\Delta}$$

$$\tan(\delta) = \frac{cw}{-m\omega^2 + k}$$

$$R^2 = \frac{F_0^2((-m\omega^2 + k)^2 + c^2\omega^2)}{\Delta^2}$$

$$= \frac{F_0^2}{\Delta}$$

$$R = \frac{F_0}{\sqrt{\Delta}}$$

2.8.2 Transient and steady-state solution

If r_1 or r_2 have positive real part, then solutions blow up. The case when r_1 or r_2 have zero real part does not arise in real-life systems. Thus, we can consider r_1 and r_2 to have negative real part (whether or not they are complex).

In this case, the solution consists of a transient part which is being killed off exponentially and a steady-state part which is the solution in the limit of large t.
The transient part is being killed off because $Re(r_1, r_2) < 0$.

2.8.3 Steady state solution

When $c = 0$, the steady state part goes as

$$u \sim \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

What happens if $\omega = \omega_0$? In this case, the solution also gives rise to resonances and beats.

$$\begin{aligned}
\text{As } t \to \infty,\ u(t) \to R\cos(\omega t - \delta) &= u_p \\
&= \frac{F_0}{\sqrt{m^2\omega^2 - 2km\omega^2 + c^2\omega^2 + k^2}} \cos(\omega t - \delta) \\
\text{When } c = 0,\ \delta = 0 \\
u(t) &\to \frac{F_0}{\sqrt{m^2\omega^4 - 2km\omega^2 + k^2}} \cos(\omega t) \\
&= \frac{F_0}{\sqrt{(-m\omega^2 + k)^2}} \cos(\omega t) \\
&= \frac{F_0}{-m\omega^2 + k} \cos(\omega t) \\
&= \frac{F_0}{m(k/m - \omega^2)} \cos(\omega t)
\end{aligned}$$

If $\omega = \sqrt{k/m}$, the solution blows up. However, in the original equation

$$mr^2 + k = 0 \to r = \pm\sqrt{\frac{k}{m}} i$$

So $\cos(\sqrt{k/m}t)$ is a fundamental solution. Therefore, we'd have a particular solution of the form $t\cos(\omega t)$.

2.9 Reduction of Order

2.9.1 Idea

If one is given a solution y_1 to

$$y'' + p(x)y' + q(x) = 0$$

one can use it to find a second solution y_2. The method is called reduction of order.

2.9.2 Form of second solution

Start by writing

$$y = u(x)y_1(x)$$

and now differentiate to get

$$\begin{aligned}
y' &= uy_1' + u'y_1 \\
y'' &= uy_1'' + 2u'y_1' + u''y_1
\end{aligned}$$

2.9.3 Solving for u

If we substitute into

$$y'' + p(x)y' + q(x)y = 0$$

we get

$$uy_1'' + 2u'y_1' + u''y_1 + p(uy_1' + u'y_1) + quy_1 = 0$$

We collect terms to get

$$u(y_1'' + py_1' + qy_1) + u'(2y_1' + py_1) + u''y_1 = 0$$

But the first term is zero since y_1 is a solution of the equation. Thus,

$$u'' + \left(p + 2\frac{y_1'}{y_1}\right)u' = 0$$

But this is a linear equation for u' and we can solve it.
Specifically, we use the integrating factor

$$\begin{aligned}
f(x) &= e^{\int p + 2\frac{y_1'}{y_1} dx} \\
&= e^{\int p dx + 2\ln y_1} \\
&= y_1^2 e^{\int p dx} \\
\frac{d}{dx}(y_1^2 e^{\int p dx} u') &= 0 \\
y_1^2 e^{\int p dx} u' &= c_1 \\
u' &= \frac{c_1}{y_1^2 e^{\int p dx}} \\
u &= \int \frac{c_1}{y_1^2 e^{\int p dx}} dx + c_2
\end{aligned}$$

Example. Verify that $y = x^{-1}$ is a solution to $x^2y'' - 2xy' - 4y = 0$ and use reduction of order to find a second solution.

$$\begin{aligned}
y' &= -x^{-2} \\
y'' &= 2x^{-3} \\
\therefore 2x^{-1} + 2x^{-1} - 4x^{-1} &= 0
\end{aligned}$$

Thus $y = x^{-1}$ is a solution.

Now we find another solution using reduction of order:

$$\begin{aligned}
y &= vy_1 \\
y' &= vy_1' + v'y_1 \\
y'' &= vy_1'' + 2v'y_1' + v''y_1 \\
0 &= x^2vy_1'' + 2x^2v'y_1' + x^2v''y_1 - 2xvy_1' - 2xv'y_1 - 4vy_1 \\
&= x^2y_1v'' + (2x^2y_1' - 2xy_1)v' \\
0 &= v'' + \left(\frac{2y_1'}{y_1} - \frac{2}{x}y_1\right)v' \\
&= v'' + \left(-\frac{2}{x} - \frac{2}{x}\right)v' \\
0 &= v'' - \frac{4}{x}v'
\end{aligned}$$

The integrating factor is

$$I = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = x^{-4}$$

We thus have

$$\begin{aligned}\frac{d}{dx}\left(x^{-4}v'\right) &= 0\\ x^{-4}v' &= c_1\\ v' &= c_1x^4\\ v &= \frac{1}{5}c_1x^5 + c_2\end{aligned}$$

Thus the solution is

$$\begin{aligned}y &= vy_1\\ &= \left(\frac{1}{5}c_1x^5 + c_2\right)x^{-1}\\ &= \frac{1}{5}c_1x^4 + c_2x^{-1}\end{aligned}$$

Hence the second fundamental solution is x^4.

3 Higher-Order Linear Differential Equations

3.1 Systems

Consider a system

$$Ax = b$$

where A is a matrix and b is a vector and we wish to solve for a vector x. This is a compact way of writing a system of linear equations. We will see presently how we can use the same notation for systems of linear differential equations. Here

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{is an } m \times n \text{ matrix.}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{is an } n\text{-dimensional vector}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{is an } m\text{-dimensional vector}$$

$Ax = b$ is equivalent to writing

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned}$$

3.2 Eigenvectors

The eigenvectors of a matrix are non-zero vectors which are sent to multiples of themselves when the matrix acts on them, i.e.

$$Ax = y = \lambda x$$

where λ is some parameter. The value of λ corresponding to an eigenvector is called an eigenvalue.
x is the eigenvector, λ is the eigenvalue whenever $Ax = \lambda x$ holds for a given A.

3.3 Finding an eigenvector

Given a matrix A, how do we find eigenvectors, that is, vectors x such that

$$Ax = \lambda x$$

for some λ?
Rewrite this as

$$Ax - \lambda x = (A - \lambda I)x = 0$$

This is a homogeneous linear equation and so it has a non-zero solution if and only if

$$|A - \lambda I| = \det(A - \lambda I) = 0$$

Otherwise, if $\det(A - \lambda I) \neq 0$, $A - \lambda I$ is invertible, so $x = 0$ is the only solution - but remember, non-zero vectors are not eigenvectors so this doesn't count.

Example. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

We need to solve $|A - \lambda I| = 0$:

$$\det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = 0$$
$$\lambda^2 - 4\lambda + 4 = 0$$

and so there are two roots (which are the same)

$$\lambda_1 = \lambda_2 = 2$$

The number of times that an eigenvalue is repeated is called its multiplicity. Here the multiplicity is 2. To find the eigenvectors we solve $(A - \lambda I)x = 0$. For $\lambda = 2$ we have

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Solve this to get an eigenvector

$$x^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and so we have only one linearly independent eigenvector.

Example. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

We have

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{pmatrix} \\ &= (1-\lambda) \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix} \\ &= (1-\lambda)(1-\lambda)(-1-\lambda) \end{aligned}$$

$\lambda_1 = -1$ has multiplicity 1 and $\lambda_2 = 1$ has multiplicity 2.
When $\lambda = 1,\ (A - \lambda I)x = 0$ is

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $x_2 = x_3 = 0$ and x_1 is an arbitrary number. So $x = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for any $\alpha \neq 0$ are the eigenvectors corresponding to $\lambda = 1$.

When $\lambda = -1,\ (A - \lambda I)x = 0$ is

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 2x_2 + x_3 &= 0 \\ x_1 &= \beta \qquad \text{(arbitrary)} \\ \rightarrow x_2 &= 2\beta \\ \rightarrow x_3 &= -4\beta \end{aligned}$$

Therefore, $x = \begin{pmatrix} \beta \\ 2\beta \\ -4\beta \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$, $\beta \neq 0$ are the eigenvectors corresponding to $\lambda = -1$.

3.4 Self-adjoint or Hermitian matrices

3.4.1 Definition

$$\bar{A}^T \equiv A^* = A$$

i.e. If you take the complex conjugate of each element of the matrix and then the transpose you get the original matrix back.

Example.

$$A = \begin{bmatrix} 3 & 2+i \\ 2-i & -4 \end{bmatrix}$$

Check:

$$\bar{A} = \begin{bmatrix} 3 & 2-i \\ 2+i & -4 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 3 & 2+i \\ 2-i & -4 \end{bmatrix}$$

3.4.2 Properties of Hermitian matrices

1. All the eigenvalues are real.

2. There always exists a full set of n linearly independent eigenvectors.

3. If $x^{(1)}$ and $x^{(2)}$ are eigenvectors corresponding to two different eigenvalues, then their inner product is zero:

$$\left(x^{(1)}, x^{(2)}\right) = 0$$

   where the inner product is defined

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$$

   The inner product in this case is just the complex dot product.

4. Corresponding to an eigenvalue of multiplicity m, it is impossible to choose m eigenvectors that are mutually orthogonal.

3.5 Homogeneous linear equations

3.5.1 Homogeneous systems

A homogeneous system of linear equations can be written $Ax = 0$.

3.5.2 Linearity

If u and v are solutions of $Ax = 0$ then a linear combination $\alpha u + \beta v$ is also a solution. This is easy to see:

$$A(\alpha u + \beta v) = \alpha Au + \beta Av = \alpha \cdot 0 + \beta \cdot 0 = 0$$

by the distribution properties of matrix multiplication. We know that $Au = 0$ and $Av = 0$ because u and v are solutions.

3.5.3 Basis

A basis of solutions of a homogeneous n-dimensional linear system

$$\vec{y}' = A\vec{y} \tag{6}$$

on some interval J is a linearly independent set of n solutions $\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots, \vec{y}^{(n)}$.

A linear combination

$$\vec{y} = c_1\vec{y}^{(1)} + c_2\vec{y}^{(2)} + \cdots + c_n\vec{y}^{(n)}$$

is called a general solution of (6) on J because this $\vec{y}$ includes every solution of (6). If the entries of A, $a_{jk}(t)$, are continuous, then (6) has a basis of solutions on J.

The Wronskian of $\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots, \vec{y}^{(n)}$ is

$$W(\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots, \vec{y}^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{vmatrix}$$

The columns are these solutions, each written in terms of components. The solutions form a basis on J if and only if W is not zero at any t_1 in this interval. W is either identically zero or nowhere zero in J.

3.6 Homogeneous Linear Systems with Constant Coefficients

3.6.1 System

We consider systems of the form

$$\frac{d}{dt}x = Ax$$

where x is a n-dimensional vector and A is an $n \times n$ matrix. This is really

$$\frac{dx(t)}{dt} = Ax(t)$$

3.6.2 Solutions

3.6.3 Equilibrium solution

One can see that $x = 0$ is a solution. This is called an equilibrium solution since then $\frac{dx}{dt}$ is always zero. Recall that an equilibrium solution is not changing with time.

3.6.4 General solution

We take inspiration from the one-dimensional case and suppose that there is a solution of the form

$$x = Fe^{\lambda t}$$

Then taking the derivative gives

$$\frac{d}{dt}x = \lambda F e^{\lambda t} = AFe^{\lambda t}$$

and so

$$AF = \lambda F$$

and we are now looking for eigenvectors and eigenvalues:

$$(A - \lambda I)F = 0$$

Here $\vec{F}$ is a vector of the same length as $\vec{x}$, independent of t.

Example. Solve

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$$

The differential equations represented by the matrix $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ are

$$x_1' = x_1 + x_2$$
$$x_2' = 4x_1 + x_2$$

We find the eigenvalues

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 1 - 2\lambda + \lambda^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$
$$\to \lambda_1 = 3$$
$$\to \lambda_2 = -1$$

We find the eigenvector for $\lambda_1 = 3$:

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$-2f_1 + f_2 = 0 \qquad \text{(\underline{both} equations)}$$
$$f_1 = \alpha$$
$$f_2 = 2\alpha$$
$$f = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\alpha, \qquad \alpha \in \mathbb{R}, \alpha \neq 0$$

Therefore, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector (we only need one).

We find the eigenvector for $\lambda_2 = -1$:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$2f_1 + f_2 = 0$$
$$f_1 = \beta$$
$$f_2 = -2\beta$$
$$f = \begin{pmatrix} 1 \\ -2 \end{pmatrix}\beta, \qquad \beta \in \mathbb{R}, \beta \neq 0$$

Therefore, $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector.

We have $x = \vec{F}e^{\lambda t}$, so

$$x^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$$
$$x^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

Note that

$$\begin{aligned} \det\left(x^{(1)}(t) \quad x^{(2)}(t)\right) &= \det\begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \\ &= -2e^{2t} - 2e^{2t} = -4e^{2t} \neq 0 \text{ ever} \end{aligned}$$

so these two solutions are linearly independent.

Thus the general solution is

$$x = c_1 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

Example. Solve

$$x' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x$$

We have

$$\begin{aligned} \det\begin{pmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{pmatrix} &= 6 + 3\lambda + 2\lambda + \lambda^2 - 2 \\ &= \lambda^2 + 5\lambda + 4 \\ &= (\lambda+4)(\lambda+1) \\ \lambda &= -4, -1 \end{aligned}$$

We find the eigenvector for $\lambda = -4$:

$$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{aligned} f_1 + \sqrt{2} f_2 &= 0 \qquad \text{(both equations)} \\ f_2 &= \alpha, \qquad \alpha \neq 0 \\ f_1 &= -\sqrt{2}\alpha \end{aligned}$$

Therefore, $f = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ is one eigenvector.

Therefore, $x^{(1)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$.

We find the eigenvector for $\lambda = -1$:

$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{aligned} \sqrt{2} f_1 - f_2 &= 0 \\ f_1 &= \beta, \qquad \beta \neq 0 \\ f_2 &= \sqrt{2}\beta \end{aligned}$$

Therefore, $f = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ is one eigenvector.

Therefore, $x^{(2)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$.

The general solution is

$$x(t) = A \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} + B \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$$

Example. Solve

$$x' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} x$$

We have

$$\begin{aligned} \det \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} &= 5 - 5\lambda - \lambda + \lambda^2 + 3 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \\ \lambda &= 4, 2 \end{aligned}$$

We find the eigenvector for $\lambda = 4$:

$$\begin{aligned} \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ f_1 - f_2 &= 0 \\ f_1 &= f_2 \end{aligned}$$

Therefore, $f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is one eigenvector.

Therefore, $x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$.

We find the eigenvector for $\lambda = 2$:

$$\begin{aligned} \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 3f_1 - f_2 &= 0 \\ f_2 &= 3f_1 \end{aligned}$$

Therefore, $f = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is one eigenvector.

Therefore, $x^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$.

The general solution is

$$x = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + B \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$$

3.7 Higher-order equations as systems of equations

3.7.1 Idea

We take higher-order equations and by renaming functions and their derivatives we can turn them into systems of first-order equations.

Example. Rewrite

$$x''' + x'' + x' + x = 0$$

as a matrix system.

This is a third-order equation so we can turn it into a system of three first-order equations. Consider the very obvious equation

$$\frac{d}{dt}\begin{pmatrix} x \\ x' \\ x'' \end{pmatrix} = \begin{pmatrix} x' \\ x'' \\ x''' \end{pmatrix}$$

Now, if we rename according to

$$x_1 = x$$
$$x_2 = x'$$
$$x_3 = x''$$

we have

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x''' \end{pmatrix}$$

We didn't have a new name for x''' because we want only a three dimensional system. We recall the original equation that we started out with

$$x''' + x'' + x' + x = 0$$

We can rearrange to get

$$x''' = -x - x' - x''$$

which we can also write as

$$x''' = -x_1 - x_2 - x_3$$

and so the system can be written

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ -x_1 - x_2 - x_3 \end{pmatrix}$$

Now this is a system of linear ODEs like the ones we have been discussing. Specifically, this is

$$\begin{array}{lclll} x_1' & = & & x_2 & \\ x_2' & = & & & x_3 \\ x_3' & = & -x_1 & -x_2 & -x_3 \end{array} \quad \text{or} \quad \vec{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \vec{x}$$

which we can solve in the usual way (although it may be much harder to find eigenvalues in this example).

3.8 Complex Eigenvalues

Let's run through the same sort of analysis that we did before and see what happens when we have complex eigenvalues. Consider

$$x' = Ax$$

where x is a vector and A is a real valued matrix.
We look for solutions of the form

$$x = Fe^{\lambda t}$$

We find the eigenvalues by setting

$$|A - \lambda I| = 0$$

and the eigenvectors by solving

$$(A - \lambda I)F = 0$$

Since A is real, any complex eigenvalues must occur in conjugate pairs. For example, if there is an eigenvalue

$$\lambda_1 = \alpha + i\beta$$

then there has to be a second eigenvalue of the form

$$\lambda_2 = \alpha - i\beta$$

Suppose that the eigenvector $F^{(1)}$ that corresponds to the eigenvalue λ_1 is of the form

$$F^{(1)} = a + ib$$

where a and b are both real vectors. Then one solution is

$$x = (a + ib)e^{(\alpha + i\beta)t}$$

and so

$$\begin{aligned} x &= (a + ib)e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \\ &= a\cos(\beta t)e^{\alpha t} - b\sin(\beta t)e^{\alpha t} + ia\sin(\beta t)e^{\alpha t} + ib\cos(\beta t)e^{\alpha t} \end{aligned}$$

We can get two solutions by taking real and imaginary parts. It might not be obvious but $\det(A - \lambda I) = 0$ is always a polynomial of degree n (where A is an $n \times n$ matrix). To solve an arbitrary polynomial, we need complex numbers because we can then find exactly n roots over the complex field.

Example. Solve

$$x' = -y, \qquad y' = x$$

We find the eigenvalues:

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \\ &= \lambda^2 + 1 = 0 \\ \lambda &= \pm i \end{aligned}$$

We find the eigenvector for $\lambda_1 = i$:

$$\begin{aligned} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ f_1 - if_2 &= 0 \qquad \text{(both equations)} \\ f_1 &= if_2 \end{aligned}$$

Therefore, $f = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector.

We find the eigenvector for $\lambda_2 = -i$:

$$\begin{aligned} \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ f_1 + if_2 &= 0 \\ f_1 &= -if_2 \end{aligned}$$

Therefore, $f = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector.

The general solution is

$$x = A \begin{pmatrix} i \\ 1 \end{pmatrix} e^{it} + B \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-it}$$

But suppose we want two real solutions? (Since A is real, this seems reasonable.) Two independent solutions are

$$\vec{u} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{it} \qquad \text{and} \qquad \vec{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-it}$$

Note that $\vec{u}$ and $\vec{v}$ are independent because

$$\det \begin{pmatrix} ie^{it} & -ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix} = i + i \neq 0$$

We can rearrange $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}
\vec{u} &= \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] (\cos(t) + i\sin(t)) \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) + i \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right] \\
\vec{v} &= \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] (\cos(t) - i\sin(t)) \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin(t) + i \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right] \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) - i \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right]
\end{aligned}$$

Any linear combination of $\vec{u}$ and $\vec{v}$ is also a solution so

$$\frac{1}{2}(\vec{u} + \vec{v}) = Re(\vec{u}) \qquad \text{and} \qquad \frac{1}{2}(\vec{u} - \vec{v}) = Im(\vec{u})$$

are also solutions.

Thus two independent real solutions are

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t)$$

Example. Solve

$$\begin{aligned}
x' &= -x - y & \qquad x(0) &= 3 \\
y' &= 4x - y & \qquad y(0) &= 4
\end{aligned}$$

Find the eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} -1-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix} \\
&= (-1-\lambda)^2 + 4 \\
&= \lambda^2 + 2\lambda + 5 \\
\lambda &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\
&= -1 \pm 2i
\end{aligned}$$

Find the eigenvector corresponding to $\lambda_1 = -1 + 2i$:

$$A - \lambda_1 I = \left(\begin{array}{cc|c} -2i & -1 & 0 \\ 4 & -2i & 0 \end{array}\right) \xrightarrow{R_2 - 2iR_1} \left(\begin{array}{cc|c} -2i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-2iv_1 - v_2 = 0$$

$$v_1 = s, \qquad v_2 = -2is$$

$$\vec{v_1} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} s, \qquad s \neq 0$$

$$\therefore \ \vec{x_1} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{(-1+2i)t} \qquad \text{is a complex solution}$$

$$= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -2 \end{pmatrix}\right] e^{-t}(\cos(2t) + i\sin(2t))$$

$$= e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin(2t)\right] + ie^{-t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos(2t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(2t)\right]$$

But $Re(\vec{x_1})$ and $Im(\vec{x_1})$ are themselves solutions.

$$\therefore \ \vec{w_1} = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin(2t)\right] \qquad \text{is a solution}$$

$$\vec{w_2} = e^{-t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos(2t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(2t)\right] \qquad \text{is a solution}$$

Therefore we have two solutions which is enough (no need to do $\lambda = -1 - 2i$). A general solution is thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin(2t)\right] + c_2 e^{-t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos(2t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(2t)\right]$$

Finally, we need to substitute the initial conditions:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$c_1 = 3, \qquad -2c_2 = 4 \rightarrow c_2 = -2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = 3e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin(2t)\right] - 2e^{-t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos(2t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(2t)\right]$$

$$= 3e^{-t}\begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix} - 2e^{-t}\begin{pmatrix} \sin(2t) \\ -2\cos(2t) \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{-t}\cos(2t) - 2e^{-t}\sin(2t) \\ 6e^{-t}\sin(2t) + 4e^{-t}\cos(2t) \end{pmatrix}$$

3.9 Nonhomogeneous higher-order equations

We can write n nth order nonhomogeneous linear differential equation as

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \tag{7}$$

where $r(x) \not\equiv 0$. To study this, we need to study the corresponding homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \tag{8}$$

just like we did for second-order equations.

Theorem 3.1. *1. The difference between any two solutions of* (7) *is a solution of* (8).

2. The sum of a solution of (7) *and a solution of* (8) *is a solution of* (7).

Proof. 1. Let u and v be two solutions of (7). Then

$$\begin{aligned}
u^{(n)} + p_{n-1}(x)u^{(n-1)} + \cdots + p_1(x)u' + p_0(x)u &= r(x) \\
v^{(n)} + p_{n-1}(x)v^{(n-1)} + \cdots + p_1(x)v' + p_0(x)v &= r(x) \\
u^{(n)} - v^{(n)} + p_{n-1}(x)u^{(n-1)} - p_{n-1}(x)v^{(n-1)} + \cdots + p_1(x)u' - p_1(x)v' + p_0(x)u - p_0(x)v &= r(x) - r(x) = 0 \\
(u-v)^{(n)} + p_{n-1}(x)(u-v)^{(n-1)} + \cdots + p_1(x)(u-v)' + p_0(x)(u-v) &= 0
\end{aligned}$$

2. Let u be a solution of (7) as above and w be a solution of (8), so that

$$w^{(n)} + p_{n-1}(x)w^{(n-1)} + \cdots + p_1(x)w' + p_0(x)w = 0$$

Then

$$\begin{aligned}
u^{(n)} + w^{(n)} + p_{n-1}(x)u^{(n-1)} + p_{n-1}(x)w^{(n-1)} + \cdots + p_1(x)u' + p_1(x)w' + p_0(x)u + p_0(x)w &= r(x) + 0 \\
(u+w)^{(n)} + p_{n-1}(x)(u+w)^{(n-1)} + \cdots + p_1(x)(u+w)' + p_0(x)(u+w) &= r(x)
\end{aligned}$$

□

Example. Consider the equation

$$y'' - 4y' + 3y = 10e^{-2x}.$$

Two solutions are

$$\begin{aligned}
u &= e^x + e^{3x} + \frac{2}{3}e^{-2x} \\
v &= 5e^x - 4e^{3x} + \frac{2}{3}e^{-2x}
\end{aligned}$$

while

$$w = -7e^x + 9e^{3x}$$

satisfies the homogeneous equation $y'' - 4y' + 3y = 0$. Show that $u - v$ satisfies the homogeneous differential equation, while $u + w$ satisfies the original differential equation.

$$\begin{aligned}
u - v &= -4e^x + 5e^{3x} \\
(u-v)' &= -4e^x + 15e^{3x} \\
(u-v)'' &= -4e^x + 45e^{3x} \\
(u-v)'' - 4(u-v)' + 3(u-v) &= (-4e^x + 45e^{3x}) - 4(-4e^x + 15e^{3x}) + 3(-4e^x + 5e^{3x}) \\
&= 0
\end{aligned}$$

Next

$$
\begin{aligned}
u + w &= -6e^{x} + 10e^{3x} + \frac{2}{3}e^{-2x} \\
(u + w)' &= -6e^{x} + 30e^{3x} - \frac{4}{3}e^{-2x} \\
(u + w)'' &= -6e^{x} + 90e^{3x} + \frac{8}{3}e^{-2x} \\
(u + w)'' - 4(u + w)' - 3(u + w) &= (-6e^{x} + 90e^{3x} + \frac{8}{3}e^{-2x}) - 4(-6e^{x} + 30e^{3x} - \frac{4}{3}e^{-2x}) + 3(-6e^{x} + 10e^{3x} + \frac{2}{3}e^{-2x}) \\
&= 10e^{-2x}
\end{aligned}
$$

which satisfies the original equation.

3.9.1 General solution

Definition 3.2. *A general solution of the nonhomogeneous linear equation (7) on some open interval I is a solution of the form*

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ is a general solution of the nonhomogeneous equation (8) on I and $y_p(x)$ is any solution of (7) containing no arbitrary constants.

Theorem 3.3. *If the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ of (7) and $r(x)$ are continuous on some open interval I, then (7) has a general solution on I and every solution of (7) is obtained by assigning suitable values to the constants in that general solution.*

This theorem says that we always have a solution and that the solution is of the appropriate form, so long as all the functions are continuous.

3.9.2 Initial-value problems

An initial-value problem for (7) consists of (7) and n initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \ldots \qquad y^{(n-1)}(x_0) = K_{n-1}.$$

Theorem 3.4. *If the coefficients of (7) and $r(x)$ are continuous on some open interval I and $x_0 \in I$, then the initial-value problem has a unique solution on I.*

This theorem says that every initial-value problem has a unique solution, so long as all the functions are continuous.

We have methods to solve linear homogeneous equations, so determining the homogeneous part $y_h(x)$ is not difficult. The main challenge is to find a method to determine the particular solution. Fortunately, there are some useful analogues of methods we already know.

3.10 Higher-order method of undetermined coefficients

As with second-order linear equations, the method of undetermined coefficients applies to the nth order nonhomogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x)$$

so long as $r(x)$ takes certain forms where its derivative is similar to itself. We saw earlier that this applied to polynomials, trigonometric functions and exponentials.

The new complication is that, with an nth order problem, the characteristic equation of the homogeneous equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

can have multiple roots of large orders m (where $m \leq n$ of course).

3.10.1 Rules for the method of undetermined coefficients

- Basic Rule: If $r(x)$ is one of the terms in the first column of:

Term in $r(x)$	Choice for y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \ldots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ and/or $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ and/or $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

  then choose the corresponding function y_p in the second column.

- Sum Rule: If $r(x)$ is a sum of any functions in the first column of the table, choose y_p to be the sum of the corresponding functions in the second column.

- Modification Rule: If a term in your choice of y_p is a solution of the homogeneous equation, then multiply y_p by x^k, where k is the smallest positive integer such that no term of $x^k y_p$ is a solution of the homogeneous equation.

Example. Solve

$$y^{(iv)} - y = 2.5e^{-3x}$$

The characteristic equation is $\lambda^4 - 1 = 0$ which has roots ± 1, $\pm i$. The general solution is thus

$$\begin{aligned} y_h &= c_1 e^x + c_2 e^{-x} + c_3 e^{ix} + c_4 e^{-ix} \\ &= c_1 e^x + c_2 e^{-x} + A \cos x + B \sin x \end{aligned}$$

We don't need the modification rule, so we can let $y_p = Ce^{-3x}$. Then

$$\begin{aligned} y_p' &= -3Ce^{-3x} \\ y_p'' &= 9Ce^{-3x} \\ y_p''' &= -27Ce^{-3x} \\ y_p^{(iv)} &= 81Ce^{-3x}. \end{aligned}$$

Thus

$$y_p^{(iv)} - y_p = 81Ce^{-3x} - Ce^{-3x} = 80Ce^{-3x} = 2.5e^{-3x}$$

and hence $C = \frac{1}{32}$. Hence

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^x + c_2 e^{-x} + A \cos x + B \sin x + \frac{1}{32} e^{-3x} \end{aligned}$$

Example. Solve

$$y''' - 3y'' + 3y' - y = 54e^x$$

The characteristic equation is $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$ which has a triple root $\lambda = 1$, so the general solution is

$$y_h = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

Thus, by the modification rule, we need to choose a higher-order particular solution. The next available order is third, so we can choose

$$y_p = Cx^3 e^x.$$

Then

$$
\begin{aligned}
y_p' &= Cx^3e^x + 3Cx^2e^x \\
y_p'' &= (Cx^3e^x + 3Cx^2e^x) + (3Cx^2e^x + 6Cxe^x) \\
&= Cx^3e^x + 6Cx^2e^x + 6Cxe^x \\
y_p''' &= (Cx^3e^x + 3Cx^2e^x) + (6Cx^2e^x + 12Cxe^x) + (6Cxe^x + 6Ce^x) \\
&= Cx^3e^x + 9Cx^2e^x + 18Cxe^x + 6Ce^x
\end{aligned}
$$

We thus have

$$
\begin{aligned}
y_p''' - 3y_p'' + 3y_p' - y_p = Cx^3e^x + 9Cx^2e^x + 18Cxe^x + 6Ce^x - 3(Cx^3e^x + 6Cx^2e^x + 6Cxe^x) \\
+ 3(Cx^3e^x + 3Cx^2e^x) - (Cx^3e^x) \\
= 0 \cdot Cx^3e^x + 0 \cdot Cx^2e^x + 0 \cdot Cxe^x + 6Ce^x = 54e^x
\end{aligned}
$$

and so $C = 9$. The solution is thus

$$
y = c_1e^x + c_2xe^x + c_3x^2e^x + 9x^3e^x.
$$

Example. Solve the initial-value problem

$$
y''' - 3y'' - 2y' + 6y = 3x^2 - 2x - 4, \qquad y(0) = 1, y'(0) = 1, y''(0) = 4.
$$

The characteristic equation is

$$
\begin{aligned}
\lambda^3 - 3\lambda^2 - 2\lambda + 6 &= 0 \\
\lambda^2(\lambda - 3) - 2(\lambda - 3) &= 0 \\
(\lambda^2 - 2)(\lambda - 3) &= 0 \\
\lambda &= \pm\sqrt{2}, 3
\end{aligned}
$$

The general solution is thus

$$
y_h = c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x} + c_3e^{3x}
$$

Next we need to find a particular solution. We guess one of the form

$$
y_p = Ax^2 + Bx + C
$$

Then

$$
\begin{aligned}
y_p' &= 2Ax + B \\
y_p'' &= 2A \\
y_p''' &= 0
\end{aligned}
$$

Thus

$$
\begin{aligned}
y_p''' - 3y_p'' - 2y_p' + 6y_p &= 0 - 3(2A) - 2(2Ax + B) + 6(Ax^2 + Bx + C) \\
&= 6Ax^2 + (6B - 4A)x + 6C - 2B - 6A \\
&= 3x^2 - 2x - 4
\end{aligned}
$$

It follows that

$$\begin{aligned}
6A &= 3 \\
\Rightarrow A &= \frac{1}{2} \\
6B - 4A &= -2 \\
\Rightarrow B - 2 &= -2 \\
B &= 0 \\
6C - 2B - 6A &= -4 \\
6C - 3 &= -4 \\
6C &= -1 \\
C &= -\frac{1}{6}.
\end{aligned}$$

Thus the solution is

$$y = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + c_3 e^{3x} + \frac{1}{2}x^2 - \frac{1}{6}.$$

Differentiating, we have

$$\begin{aligned}
y' &= \sqrt{2}c_1 e^{\sqrt{2}x} - \sqrt{2}c_2 e^{-\sqrt{2}x} + 3c_3 e^{3x} + x \\
y'' &= 2c_1 e^{\sqrt{2}x} + 2c_2 e^{-\sqrt{2}x} + 9c_3 e^{3x} + 1
\end{aligned}$$

Applying the initial conditions, we have

$$\begin{aligned}
y(0) &= c_1 + c_2 + c_3 - \frac{1}{6} = 1 \\
y'(0) &= \sqrt{2}c_1 - \sqrt{2}c_2 + 3c_3 = 1 \\
y''(0) &= 2c_1 + 2c_2 + 9c_3 + 1 = 4
\end{aligned}$$

We thus have to solve the linear system

$$\begin{aligned}
c_1 + c_2 + c_3 &= \frac{7}{6} \\
\sqrt{2}c_1 - \sqrt{2}c_2 + 3c_3 &= 1 \\
2c_1 + 2c_2 + 9c_3 &= 3
\end{aligned}$$

We have

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{7}{6} \\ \sqrt{2} & -\sqrt{2} & 3 & 1 \\ 2 & 2 & 9 & 3 \end{array}\right] &\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{7}{6} \\ 2 & -2 & 3\sqrt{2} & \sqrt{2} \\ 2 & 2 & 9 & 3 \end{array}\right] R_2 \times \sqrt{2} \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{7}{6} \\ 0 & -4 & 3\sqrt{2} - 2 & \sqrt{2} - \frac{7}{3} \\ 0 & 0 & 7 & \frac{2}{3} \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}
\end{aligned}$$

Thus

$$\begin{aligned}
c_3 &= \frac{2}{21} \\
-4c_2 + (3\sqrt{2}-2)\frac{2}{21} &= \sqrt{2} - \frac{7}{3} \\
-4c_2 &= \frac{5}{7}\sqrt{2} - \frac{15}{7} \\
c_2 &= \frac{15}{28} - \frac{5}{28}\sqrt{2} \\
c_1 &= \frac{7}{6} + \frac{5}{28}\sqrt{2} - \frac{15}{28} - \frac{2}{21} \\
&= \frac{15}{28} + \frac{5}{28}\sqrt{2}
\end{aligned}$$

It follows that the solution to the initial-value problem is

$$y = \left(\frac{15}{28} + \frac{5}{28}\sqrt{2}\right)e^{\sqrt{2}x} + \left(\frac{15}{28} - \frac{5}{28}\sqrt{2}\right)e^{-\sqrt{2}x} + \frac{2}{21}e^{3x} + \frac{1}{2}x^2 - \frac{1}{6}.$$

3.11 Higher-order method of variation of parameters

We want to solve the nth order system

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)$$

The method of variation of parameters (similar to what was done for second-order equations) is given by

$$\boxed{y_p(x) = y_1(x)\int \frac{W_1(x)}{W(x)}r(x)dx + y_2(x)\int \frac{W_2(x)}{W(x)}r(x)dx + \cdots + y_n(x)\int \frac{W_n(x)}{W(x)}r(x)dx}$$

where $y_1, y_2, \ldots, y_n$ is a basis of solutions of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,$$

W is the Wronskian and W_j $(j = 1, 2, \ldots, n)$ is obtained from$W$ by replacing the jth column of W by the n-dimensional column

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Thus when $n = 2$,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1.$$

Hence the formula matches the earlier one for second-order equations when $n = 2$.

Write the homogeneous equation as $L[y] = 0$. In a general solution of the homogeneous equation,

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

we replace the constants $c_1, c_2, \ldots, c_n$ by functions $u_1(x), u_2(x), \ldots, u_n(x)$, to be determined. We thus have

$$y_p = u_1y_1 + u_2y_2 + \cdots + u_ny_n.$$

We have to impose n conditions on the n arbitrary functions u_j. We will impose $(n-1)$ of these with a view to getting rid of as many derivatives of u_j as possible in $y_p', y_p'', \ldots$ The final condition will come from the differential equation itself.

Differentiating, we have

$$y_p' = (u_1 y_1' + \cdots + u_n y_n') + (u_1' y_1 + \cdots + u_n' y_n)$$

Thus, the first of the $(n-1)$ conditions to impose is

$$u_1' y_1 + \cdots + u_n' y_n = 0.$$

Differentiating what is left, we have

$$y_p'' = (u_1 y_1'' + \cdots + u_n y_n'') + (u_1' y_1' + \cdots + u_n' y_n').$$

We impose the second condition:

$$u_1' y_1' + \cdots + u_n' y_n' = 0$$

We continue this process until we reach

$$y_p^{(n-1)} = (u_1 y_1^{(n-1)} + \cdots + u_n y_n^{(n-1)}) + (u_1' y_1^{(n-2)} + \cdots + u_n' y_n^{(n-2)})$$

with the last of the $(n-1)$ conditions as:

$$u_1' y_1^{(n-2)} + \cdots + u_n' y_n^{(n-2)} = 0$$

Finally, we obviously want y_p to be a solution of the original differential equation. To do this, we need $y_p^{(n)}$, which we can get from differentiating $y_p^{(n-1)}$:

$$y_p^{(n)} = (u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)})$$

Substituting into the original differential equation, we have

$$\begin{aligned}
y_p^{(n)} + p_{n-1}(x) y_p^{(n-1)} + \cdots + p_1(x) y_p' + p_0(x) y_p &= r(x) \\
\left[(u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)})\right] + p_{n-1}\left[(u_1 y_1^{(n-1)} + \cdots + u_n y_n^{(n-1)})\right] & \\
+ \cdots + p_1\left[u_1 y_1' + \cdots + u_n y_n'\right] + p_0\left[u_1 y_1 + \cdots + u_n y_n\right] &= r(x) \\
u_1 L[y_1] + u_2 L[y_2] + \cdots + u_n L[y_n] + (u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}) &= r(x)
\end{aligned}$$

by rearrangement. However, $L[y_1] = L[y_2] = \cdots = L[y_n] = 0$ since these are all solutions of the homogeneous equation. This gives us the condition

$$u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = r(x).$$

Collecting everything, we thus have n equations:

$$\begin{aligned}
u_1' y_1 + \cdots + u_n' y_n &= 0 \\
u_1' y_1' + \cdots + u_n' y_n' &= 0 \\
&\vdots \\
u_1' y_1^{(n-2)} + \cdots + u_n' y_n^{(n-2)} &= 0 \\
u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} &= r(x).
\end{aligned}$$

We can write this in matrix form:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r \end{bmatrix}$$

Since $y_1, y_2, \ldots, y_n$ is a basis of solutions, the first matrix is invertible. We can thus use Cramer's rule, which says that

$$u_1' = W_1 r(x)/W, \qquad u_2' = W_2 r(x)/W, \qquad \cdots \qquad u_n' = W_n r(x)/W.$$

Finally, we solve for u_j by integrating:

$$u_1 = \int \frac{W_1}{W} r(x)dx, \qquad u_2 = \int \frac{W_2}{W} r(x)dx, \qquad \cdots \qquad u_n = \int \frac{W_n}{W} r(x)dx.$$

Example. Solve

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x$$

Notice that it is not possible to solve this using the method of undetermined coefficients, since the right hand side is not in a form that allows us to guess the particular solution.

Step 1: General solution.

Substitute $y = x^m$ and its derivatives into the homogeneous equation.

$$\begin{aligned} x^3 m(m-1)(m-2)x^{m-3} - 3x^2 m(m-1)x^{m-2} + 6xmx^{m-1} - 6x^m &= 0 \\ m(m-1)(m-2) - 3m(m-1) + 6m - 6 &= 0 \\ (m-1)[m(m-2) - 3m + 6] &= 0 \\ (m-1)[m^2 - 5m + 6] &= 0 \\ (m-1)(m-2)(m-3) &= 0 \\ m &= 1, 2, 3 \end{aligned}$$

Thus, a basis for the homogeneous system is

$$y_1 = x, \qquad y_2 = x^2, \qquad y_3 = x^3.$$

Step 2: Find the determinants, using Cramer's rule.

We have

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 12x^3 + 0 + 2x^3 - 0 - 6x^3 - 6x^3 = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = 0 + 3x^4 + 0 - 2x^4 - 0 - 0 = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = 0 + 0 + x^3 - 0 - 0 - 3x^3 = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = 2x^2 + 0 + 0 - 0 - 0 - x^2 = x^2$$

Step 3: Integration.

Crucially, we need the equation in standard form, so that the leading term has coefficient 1. This won't matter for the homogeneous solution or the determinants, but it matters enormously for the particular solution. So the equation is really

$$y''' - 3x^{-1}y'' + 6x^{-2}y' - 6x^{-3}y = x \ln x$$

We thus have

$$\begin{aligned} u_1 &= \int \frac{x^4}{2x^3} x \ln x dx \\ u_2 &= \int \frac{-2x^3}{2x^3} x \ln x dx \\ u_3 &= \int \frac{x^2}{2x^3} x \ln x dx. \end{aligned}$$

Hence

$$\begin{aligned} u_1 &= \frac{1}{2}\int x^2 \ln x dx \\ &\qquad u = \ln x \qquad v' = x^2 \\ &\qquad u' = \frac{1}{x} \qquad v = \frac{x^3}{3} \\ u_1 &= \frac{1}{6}x^3 \ln x - \frac{1}{6}\int x^2 dx \\ &= \frac{1}{6}x^3 \ln x - \frac{1}{18}x^3 \\ u_2 &= -\int x \ln x dx \\ &\qquad u = \ln x \qquad v' = x \\ &\qquad u' = \frac{1}{x} \qquad v = \frac{x^2}{2} \\ u_2 &= -\frac{1}{2}x^2 \ln x + \frac{1}{2}\int x dx \\ &= -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 \\ u_3 &= \frac{1}{2}\int \ln x dx \\ &\qquad u = \ln x \qquad v' = 1 \\ &\qquad u' = \frac{1}{x} \qquad v = x \\ u_3 &= \frac{1}{2}x \ln x - \frac{1}{2}\int 1 dx \\ &= \frac{1}{2}x \ln x - \frac{1}{2}x \end{aligned}$$

Thus the particular solution is

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= \left(\frac{1}{6}x^3 \ln x - \frac{1}{18}x^3\right) x + \left(-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2\right) x^2 + \left(\frac{1}{2}x \ln x - \frac{1}{2}x\right) x^3 \end{aligned}$$

Finally, the full solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 x + c_2 x^2 + c_3 x^3 + \left(\frac{1}{6}x^3 \ln x - \frac{1}{18}x^3\right) x \\
&\quad + \left(-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2\right) x^2 + \left(\frac{1}{2}x \ln x - \frac{1}{2}x\right) x^3
\end{aligned}
$$

Note: We could have included the constants of integration in the integrals of the u_j's. In this case, we would have

$$
\begin{aligned}
u_1 &= \frac{1}{6}x^3 \ln x - \frac{1}{18}x^3 + k_1 \\
u_2 &= -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + k_2 \\
u_3 &= \frac{1}{2}x \ln x - \frac{1}{2}x + k_3
\end{aligned}
$$

Then

$$
\begin{aligned}
y_p &= \left(\frac{1}{6}x^3 \ln x - \frac{1}{18}x^3 + k_1\right) x + \left(-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + k_2\right) x^2 \\
&\quad + \left(\frac{1}{2}x \ln x - \frac{1}{2}x + k_3\right) x^3 \\
&= k_1 x + k_2 x^2 + k_3 x^3 + \left(\frac{1}{6}x^3 \ln x - \frac{1}{18}x^3\right) x + \left(-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2\right) x^2 \\
&\quad + \left(\frac{1}{2}x \ln x - \frac{1}{2}x\right) x^3
\end{aligned}
$$

which is actually the entire solution.

The only reason we don't do this is because then y_p isn't quite a "particular" solution (since it has general constants)... But that's just a mathematical formality and there's nothing to stop you doing this if you prefer to think about it this way.

The method of variation of parameters works for general nonhomogeneous terms. Its biggest limitation is that we may not be able to integrate all the u_j's, if the functions are particularly complicated. Nevertheless, we can at the very least derive an implicit solution, so in a sense this method always works.

4 Systems of Differential Equations

4.1 Homogeneous Linear Systems with Constant Coefficients

We don't know how to solve ODEs in general. But we do know how to solve equations. This means we can usually find equilibria. Better than that, however, is that once we find equilibria, the linear approximation in a local neighbourhood of the equilibrium gives us the same stability properties as the nonlinear system.

Example. Rewrite $\begin{aligned} x' &= -x \\ y' &= x^2 + y \end{aligned}$ with a linear and a nonlinear component. Solve the linear system if the nonlinear part is ignored.

We have

$$\begin{aligned} x' &= -x & \qquad x' &= 0 \to x = 0 \\ y' &= x^2 + y & \qquad y' &= 0 \to x^2 + y = 0 \to y = 0 \textit{ since } x = 0 \end{aligned}$$

Therefore, $(0,0)$ is the only equilibrium. We can rewrite the system as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ x^2 \end{pmatrix}$$

linear part $\nearrow$ $\qquad$ $\nwarrow$ nonlinear part

Near $(0,0)$, x^2 is tiny, so we could ignore it.

$$\text{Thus, } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{Which has the solution } \begin{pmatrix} x \\ y \end{pmatrix} = \exp\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} t\right] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\text{Define } \exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

$$\text{In this case, } \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}^k = \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix}$$

$$\begin{aligned} \text{So } \exp\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} t\right] &= \exp\begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} -t^3 & 0 \\ 0 & t^3 \end{pmatrix} + \ldots \\ &= \begin{pmatrix} 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots & 0 \\ 0 & 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots \end{pmatrix} \end{aligned}$$

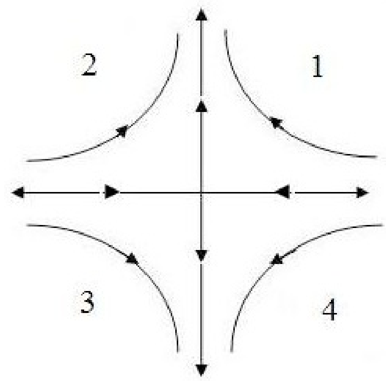
Therefore the solution to the linear part is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} e^{-t} x_0 \\ e^t y_0 \end{pmatrix}$$

$$\therefore y = y_0 e^t \text{ and } x = x_0 e^{-t} \text{ so } e^t = \frac{x_0}{x} \text{ and } y = \frac{y_0 x_0}{x} = \frac{c}{x}$$

But what direction do we have?

In quadrant 1, $x, y > 0, x' = -x < 0$
Then we can figure out the direction
of other quadrants by extrapolation
or by testing the other quadrants.



This equilibrium is a saddle and occurs when one eigenvalue is positive and the other is negative.
Therefore, $(0,0)$ is unstable, unless initial conditions start on the x-axis.

What about the nonlinear system?

$$x = x_0 e^t, \qquad y' = x_0^2 e^{-2t} + y$$
$$y' - y = x_0 e^{-2t}$$
$$\text{Now, } e^{-t}y' - e^{-t}y = x_0 e^{-3t}$$
$$\frac{d}{dt}(e^{-t}y) = x_0 e^{-3t} \qquad \text{integrating factor: } e^{-t}$$
$$\to \int_0^t d(e^{-t}y) = \int_0^t x_0^2 e^{-3t} dt \to e^{-t}y\Big|_0^t = \frac{x_0^2 e^{-3t}}{-3}\Big|_0^t$$
$$\to e^{-t}y - y_0 = \frac{x_0 e^{-3t}}{-3} + \frac{x_0^2}{3} \to e^{-t}y = \frac{x_0^2}{-3}e^{-3t} + \frac{x_0^2}{3} + y_0$$
$$\to y = \left(\frac{x_0^2}{3} + y_0\right)e^t - \frac{x_0^2}{3}e^{-2t}$$

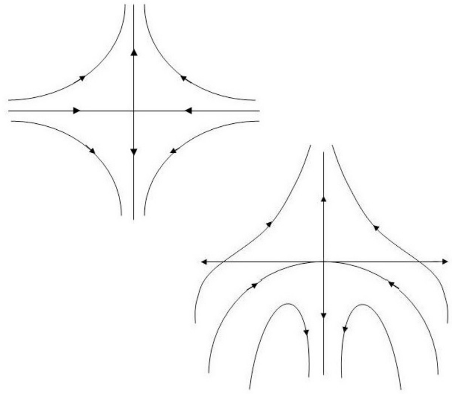
$\nearrow$ $ke^t \to$ same as linear part

$\nwarrow$ term makes no difference if x_0 is small

But remember, $x = x_0 e^{-t}$, so $e^t = \dfrac{x_0}{x}$

$$\therefore y = \left(\frac{x_0^2}{3} + y_0\right)\frac{x_0}{x} - \frac{x_0^2}{3}\left(\frac{x}{x_0}\right)^2$$
$$= \frac{x_0^2 + 3y_0 x_0 - x^3}{3x} = \frac{c - x^3}{3x}$$
$$\text{If } c = 0, y = \frac{-x^3}{3x} = -\frac{1}{3}(x_0 e^{-t})^2 = \frac{1}{3}x_0^2 e^{-2t} \to 0 \text{ as } t \to \infty$$

Therefore the nonlinear part has a distorting effect on the curves. However,
a) the behaviour isn't fundamentally different
b) locally, near the origin, the curves are similar.
So can we just ignore the nonlinear part?
Yes, almost always.

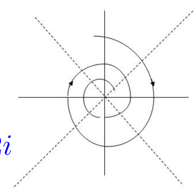


Example. Find the equilibrium for $\begin{array}{l} x' = -x + 2y + x^3 + y^2 x \\ y' = -2x - y + y^3 + x^2 y \end{array}$ and determine its stability. Draw solutions to the linear approximation in the phase plane, close to the equilibrium.

Clearly (0,0) is the only equilibrium.

Linear part: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow$ look at eigenvalues.

$$\left| \begin{pmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix} \right| = (1+\lambda)^2 + 4 = \lambda^2 + 2\lambda + 5 = 0 \rightarrow \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$



Solutions look like $e^{\lambda t} = e^{-t} e^{\pm 2it}$ i.e. $e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$.

Exercise. Find the general solution for the linear part and explain why the direction is clockwise.

Example. Sketch the full solution to the nonlinear system in the phase plane. (Hint: Transform to polar coordinates.)

Since we have $x^2 + y^2$ terms, let's transform to polar coordinates.

$$x' = -x + 2y + x^3 + y^2 x \quad (1)$$
$$y' = -2x - y + y^3 + x^2 y \quad (2)$$

In polar coordinates: $x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2$

$$\rightarrow r'\cos\theta - r\sin\theta\theta' = -r\cos\theta + 2r\sin\theta + r\cos\theta(r^2)$$
$$\rightarrow r'\sin\theta + r\cos\theta\theta' = -2r\cos\theta - r\sin\theta + r\sin\theta(r^2)$$

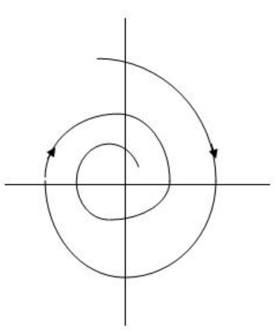
$$Equation(1): \quad \begin{array}{l} \cos\theta : r' = -r + r^3 \\ sin\theta : -r\theta' = 2r \rightarrow \theta' = -2 \end{array} \qquad Equation(2): \quad \begin{array}{l} \sin\theta : r' = -r + r^3 \\ \cos\theta : r\theta' = -2r \rightarrow \theta' = -2 \end{array}$$

$\therefore$ direction is clockwise since $\theta' < 0$

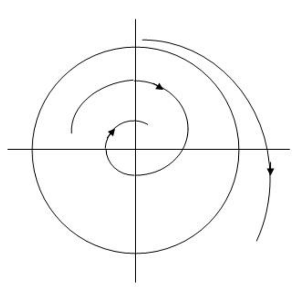
Now we are left with $r' = -r + r^3$, so $r' = 0$ when $r = 0$ or $r = 1$ ($r = -1$ doesn't make sense)

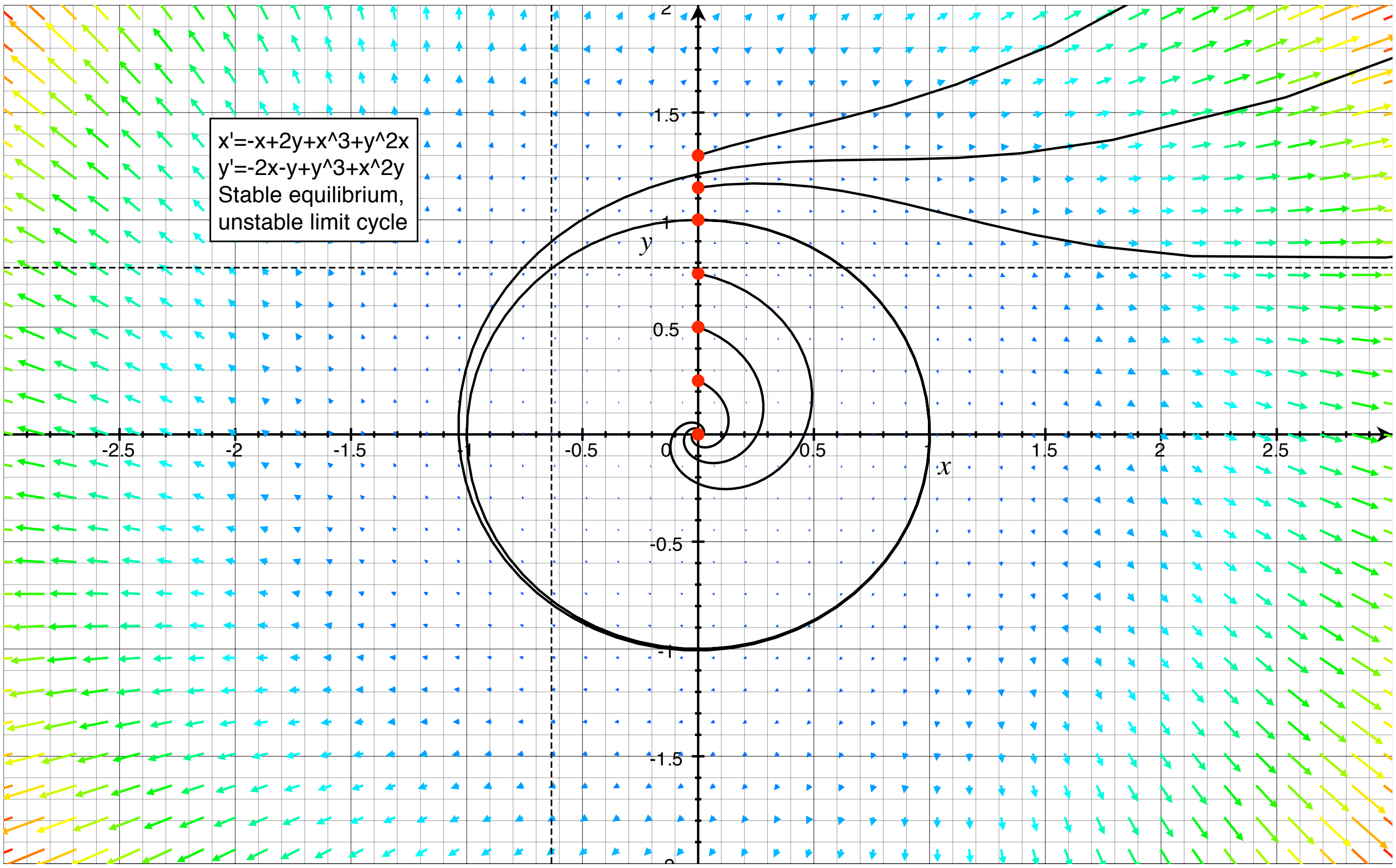
When $0 < r < 1, r' < 0$ When $r > 1, r' > 0$

Linear



Nonlinear





Therefore, in the linear system, all solutions spiralled into the origin. In the nonlinear system, solutions near the origin spiralled in, but solutions farther away spiral out. Thus, locally the behaviour of the nonlinear system is not quantifiably different from the behaviour in the linear system. In these examples, knowing the stability of the origin in the linear system is good enough for the stability of the origin in the nonlinear system. This isn't always true though.

Example. $x' = 2y + \epsilon x(x^2+y^2)$ $\quad$ ϵ is some parameter that may be positive
$y' = -2x + \epsilon y(x^2+y^2)$ $\quad$ $(0,0)$ is the only equilibrium

Linear part: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 2 \\ -2 & -\lambda \end{pmatrix} = \lambda^2 + 4 = 0 \to \lambda = \pm 2i$$

Solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \exp\left[\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} t\right]\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$\exp\begin{pmatrix} 0 & 2t \\ -2t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2t \\ -2t & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -(2t)^2 & 0 \\ 0 & -(2t)^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & -(2t)^3 \\ (2t)^3 & 0 \end{pmatrix} + \dots$$

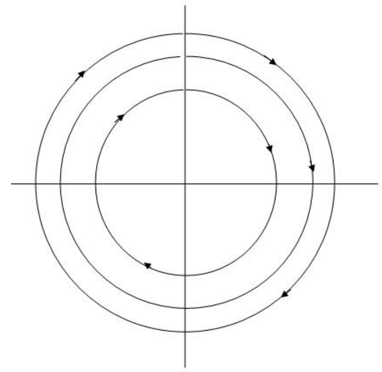
$$= \begin{pmatrix} 1 - \frac{(2t)^2}{2!} + \frac{(2t^4)}{4!} - \dots & 2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \dots \\ -2t + \frac{(2t)^3}{3!} - \frac{(2t)^5}{5!} + \dots & 1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \dots \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}$$

$$So \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0\cos(2t) + y_0\sin(2t) \\ -x_0\sin(2t) + y_0\cos(2t) \end{pmatrix}$$

$$Now\ x^2+y^2 = x_0^2\cos^2(2t) + 2x_0y_0\cos(2t)\sin(2t) + y_0^2\sin^2(2t) + x_0^2\sin^2(2t) - 2x_0y_0\cos(2t)\sin(2t) + y_0^2\cos^2(2t)$$
$$= x_0^2 + y_0^2$$

Therefore, solutions are circles with center $(0,0)$ and radius $\sqrt{x_0^2 + y_0^2}$.
The direction is clockwise.
This is called a centre.

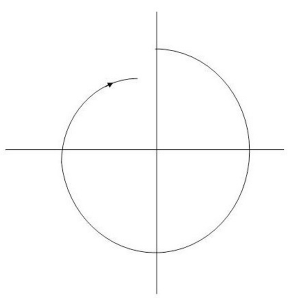


For the nonlinear part, we use polar coordinates:

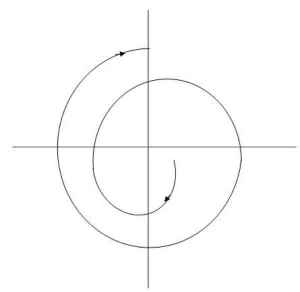
$$r' \cos\theta - r\sin\theta\theta' = 2r\sin\theta + \epsilon r\cos\theta(r^2)$$
$$r' \sin\theta + r\cos\theta\theta' = -2r\cos\theta + \epsilon r\sin\theta(r^2)$$

Thus, $r' = \epsilon r^3$ and $\theta' = -2$.

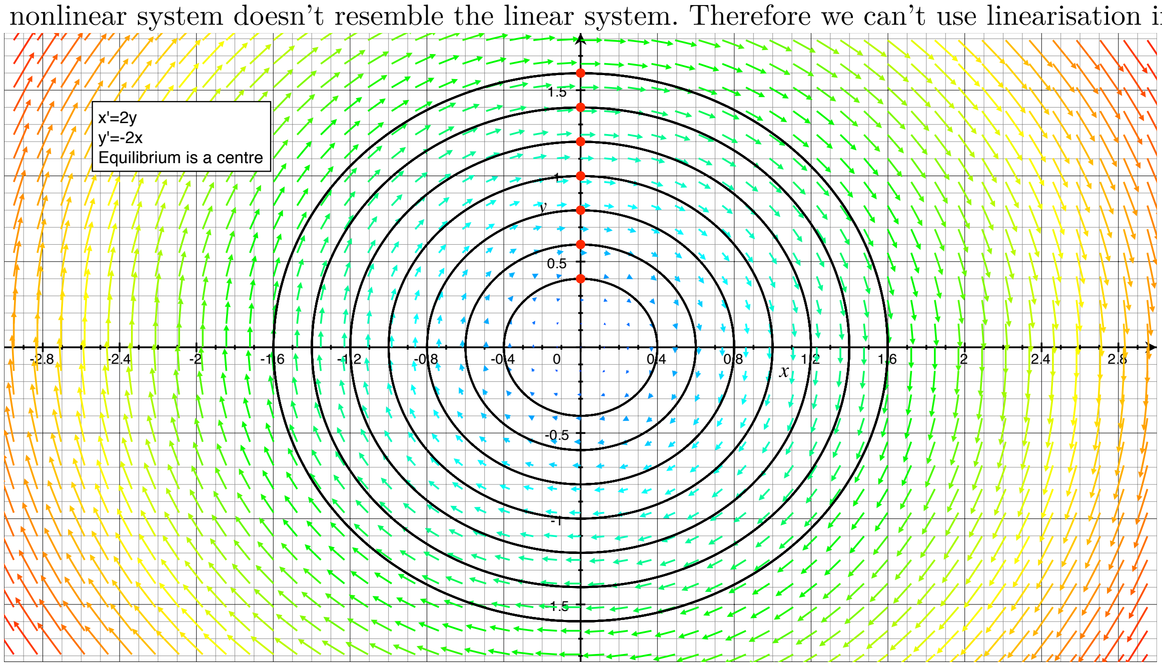
If $\epsilon < 0$, solutions spiral into the origin.



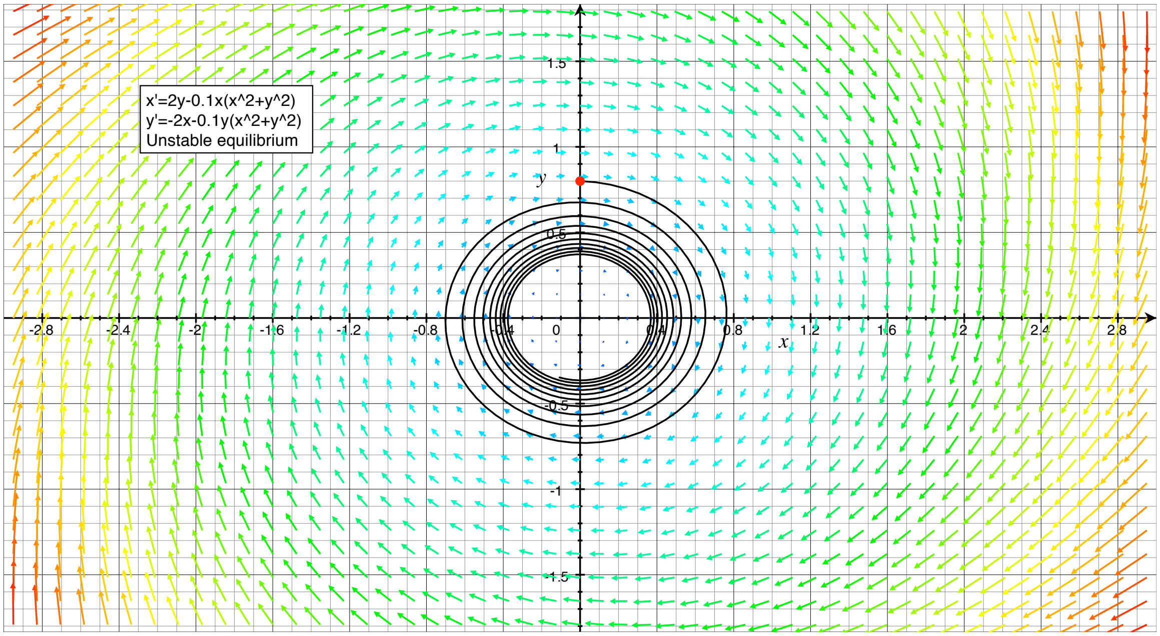
If $\epsilon > 0$, solutions spiral away from the origin.



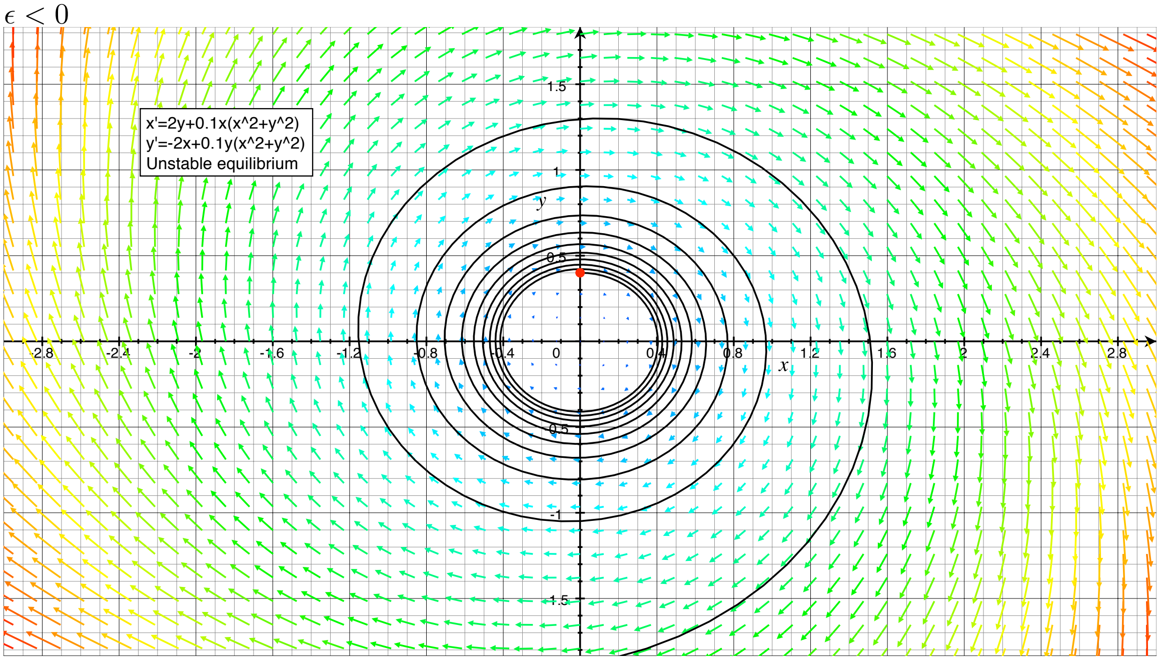
Either way, this is qualitatively different behaviour from the linear system. No matter how close to the origin, the nonlinear system doesn't resemble the linear system. Therefore we can't use linearisation in this case.



$\epsilon = 0$



$\epsilon < 0$



$\epsilon > 0$

Example. Show that the linear approximation of

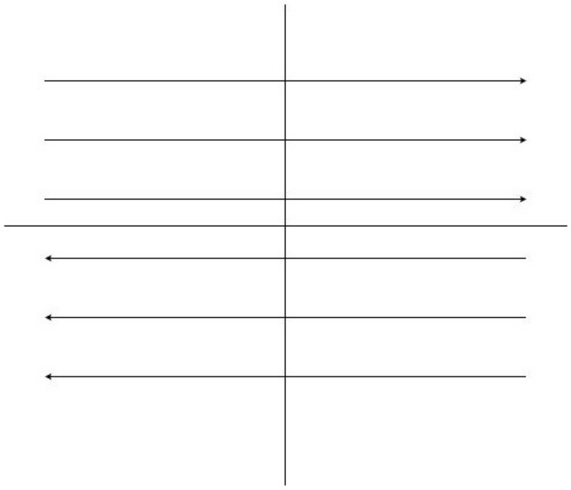
$$\begin{aligned} x' &= y \\ y' &= x^2 \end{aligned}$$

does not resemble the nonlinear behaviour.

$(0,0)$ is the only equilibrium in the full system.

Linear part is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \to y = k, x' = k \to x = kt + x_0$

However, in the linear system, $(x_0, 0)$ are equilibria.



Direction for the nonlinear part?

$$
\begin{aligned}
&(1)\ x' > 0, y' > 0\\
&(2)\ x' > 0, y' > 0\\
&(3)\ x' < 0, y' > 0\\
&(4)\ x' < 0, y' > 0
\end{aligned}
$$

We can look at $\frac{dy}{dx} = \frac{y'}{x'} = \frac{x^2}{y}$

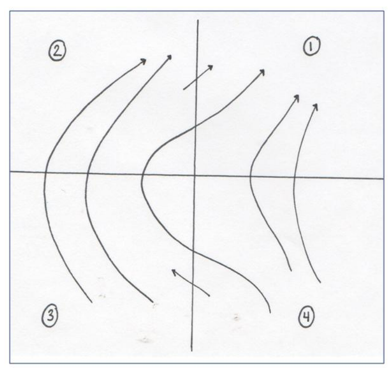
$$\rightarrow \int y dy = \int x^2 dx$$

$$\frac{y^2}{2} - \frac{y_0}{2} = \frac{x^3}{3} - \frac{y_0^3}{3}$$

$$3y^2 = 2x^3 + k$$

$$\rightarrow y = \pm\sqrt{\frac{2x^3+k}{3}}$$

$$k = 0 \rightarrow y = \pm\sqrt{\frac{2}{3}}x^{\frac{3}{2}}$$



The linearisation wasn't much help in this case: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\lambda = 0, 0.$

Therefore the behaviour of the linear and nonlinear is quite different.

If one of the eigenvalues has zero real part, then the behaviour of the nonlinear system cannot be approximated by the behaviour of the linear system. If none of the eigenvalues has zero real part, then we say the equilibrium is hyperbolic.

4.2 The Linearisation Theorem

Suppose the n-dimensional system $x' = F(x)$ has an equilibrium point at x_0 that is hyperbolic. Then the nonlinear flow is topologically equivalent to the flow of the linearised system in a neighbouhood of x_0.

Let $x' = F(x)$ and suppose $F(x_0) = 0$.

Let $J = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$ denote the Jacobian matrix of the system.

Then the linear system, $x' = Jx$ is called the linearised system near x_0. If $x_0 = 0$, the linearised system is obtained by simply dropping all the nonlinear terms in F. For hyperbolic equilibria, the eigenvalues of J completely determine the stability of x_0.

Consider $\begin{array}{l} x' = f(x,y) \\ y' = g(x,y) \end{array}$ with $\begin{array}{l} f(x_0,y_0) = 0 \\ g(x_0,y_0) = 0 \end{array}$

If we make a change of variables, $u = x - x_0$, $v = y - y_0$, then the new system has an equilibrium at $(0,0)$. So we may as well assume that $x_0 = y_0 = 0$.
Let's consider a number of possibilities when $(0,0)$ is hyperbolic:

a) Eigenvalues are real, negative and distinct: $-\lambda, -\mu < 0$, then we write:

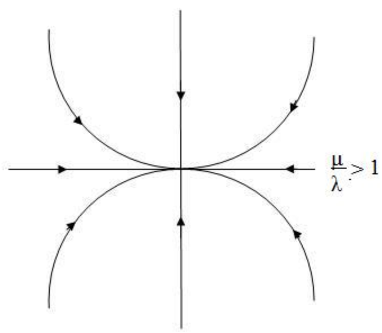
$$\begin{aligned} x' &= -\lambda x + h_1(x,y) \\ y' &= -\mu y + h_2(x,y) \end{aligned}$$

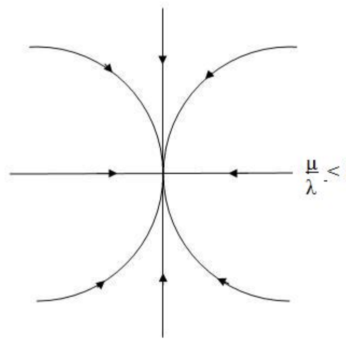
where $h_j(x,y)$ contains all higher order terms ie $h_j(x,y) = 0(x^2+y^2)$.
The linearised system is

$$\begin{array}{l} x' = -\lambda x \\ y' = -\mu y \end{array}, \quad \frac{dy}{dx} = \frac{y'}{x'} = \frac{\mu y}{\lambda x} \to \int \frac{1}{y} dy = \frac{\mu}{\lambda} \int \frac{1}{x} dx$$

$So\ \ln \frac{y}{y_0} = \frac{\mu}{\lambda} \ln \frac{x}{x_0} \to y = kx^{\frac{\mu}{\lambda}}$ (we know that $\frac{\mu}{\lambda} > 0$, but now we wonder if $\frac{\mu}{\lambda} > 1$?)

Direction?
1) $x' < 0, y' < 0$
2) $x' > 0, y' < 0$
3) $x' > 0, y' > 0$
4) $x' < 0, y' > 0$





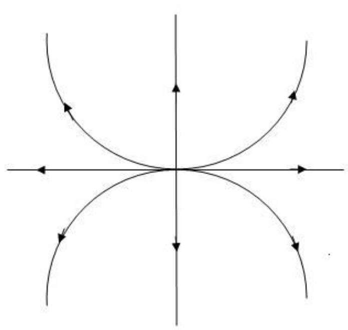
Solutions get sucked into the origin. We call this point a <u>sink.</u> By the linearisation theorem, the origin of the nonlinear system will also be a sink.

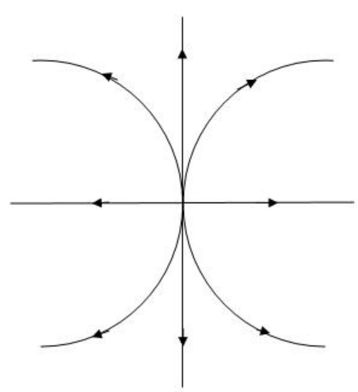
b) Eigenvalues are real, positive and distinct: $\lambda, \mu > 0$.
Then we write:

$$\begin{aligned} x' &= \lambda x + h_1(x,y) \\ y' &= \mu y + h_2(x,y) \end{aligned}$$

The linearised system is $\begin{array}{l} x' = \lambda x \\ y' = \mu y \end{array}$ $\to y = kx^{\frac{\mu}{\lambda}}$ as before

Direction?
1) $x' > 0, y' > 0$
2) $x' < 0, y' > 0$
3) $x' < 0, y' < 0$
4) $x' > 0, y' < 0$





Solutions are pushed away from the origin. We call this equilibrium a <u>source</u>. By the linearisation theorem, the origin of the nonlinear system will also be a source.

c) Eigenvalues are real, positive and equal: $\lambda, \lambda > 0$. In this case, we can't use

$$\begin{aligned} x' &= \lambda x \\ y' &= \lambda y \end{aligned}$$

as the linearised system, since the solution is $x = c_1 e^{\lambda t}, y = c_2 e^{\lambda t} = \frac{c_2}{c_1} x$ and thus x and y aren't independent. Recall from linear algebra that the answer is to include a higher order term in one variable:

$$\begin{aligned} x &= c_1 e^{\lambda t} + c_3 t e^{\lambda t} \\ y &= c_2 e^{\lambda t} \end{aligned}$$

These are clearly independent.
Check the derivatives:

$$\begin{aligned} x' &= \lambda c_1 e^{\lambda t} + c_3 e^{\lambda t} + \lambda t c_3 e^{\lambda t} = \lambda x + \tfrac{c_3}{c_2} y \\ y' &= \lambda c_2 e^{\lambda t} = \lambda y \end{aligned} \qquad \left(\begin{array}{c} \leftarrow \text{ we get this term from the } y' \text{ equation:} \\ \lambda c_2 e^{\lambda t} = \lambda y \to e^{\lambda t} = \frac{y}{c_2} \end{array} \right)$$

And the system is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda & \frac{c_3}{c_2} \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ and the eigenvalues are still λ, λ.
In this case, x and y are always increasing, so the equilibrium is a source. (ie $(x, y) \not\to (0, 0)$ as $t \to \infty$).

d) Eigenvalues are real, negative and equal: $-\lambda, -\lambda < 0$.
By symmetry, the linear system is:

$$\begin{aligned} x' &= -\lambda x + ay \\ y' &= -\lambda y \end{aligned}$$

and the solution is

$$\begin{aligned} x &= c_1 e^{-\lambda t} + c_2 t e^{-\lambda t} \\ y &= c_2 e^{-\lambda t} \end{aligned}$$

Therefore $(x, y) \to (0, 0)$ *as* $t \to \infty$ so this is a sink.

e) Eigenvalues are complex with negative real part: $-\alpha \pm i\beta, -\alpha < 0, \beta \neq 0$.
The linearised system is

$$\begin{aligned} x' &= -\alpha x + \beta y \\ y' &= -\alpha y - \beta x \end{aligned}$$

$$\textit{Since } \det \begin{pmatrix} -\alpha - \lambda & \beta \\ -\beta & -\alpha - \lambda \end{pmatrix} = (\alpha + \lambda)^2 + \beta^2 = \lambda^2 + 2\alpha\lambda + \alpha^2 + \beta^2 = 0$$

$$\textit{So } \lambda = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2 + \beta^2)}}{2} = -\alpha \pm \beta i$$

The solution is:

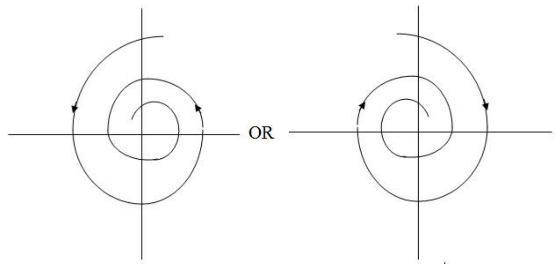
$$\begin{aligned} x &= e^{-\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) \\ y &= e^{-\alpha t}(c_2 \cos \beta t - c_1 \sin \beta t) \end{aligned}$$

$$\begin{aligned}
\text{Differentiate:}\qquad x' &= -\alpha e^{-\alpha t}(c_1\cos\beta t + c_2\sin\beta t) + e^{-\alpha t}(-c_1\beta\sin\beta t + c_2\beta\cos\beta t)\\
y' &= -\alpha e^{-\alpha t}(c_2\cos\beta t - c_1\sin\beta t) + e^{-\alpha t}(-c_2\beta\sin\beta t - c_1\beta\cos\beta t)\\
So\qquad x' &= -\alpha x + \beta y\\
y' &= -\alpha y - \beta x
\end{aligned}$$

Also note that

$$\begin{aligned}
x^2+y^2 &= e^{-2\alpha t}[c_1^2\cos^2\beta t + 2c_1c_2\cos\beta t\sin\beta t + c_2^2\sin^2\beta t\\
&\quad + c_2^2\cos^2\beta t - 2c_1c_2\cos\beta t\sin\beta t + c_1^2\sin^2\beta t]\\
&= e^{-2\alpha t}(c_1^2+c_2^2) \to 0 \;as\; t\to\infty
\end{aligned}$$

Therefore, this is a spiral sink.



f) Eigenvalues are complex with positive real part: $\alpha \pm i\beta$.
By symmetry, the linearised system is

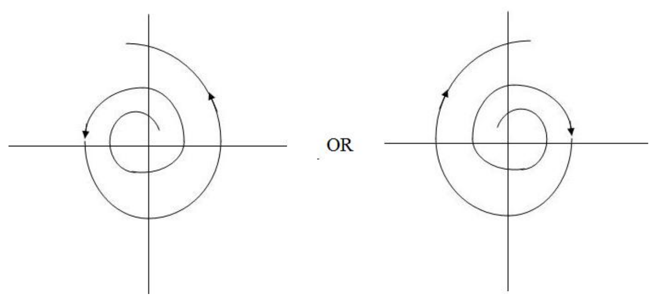
$$\begin{aligned}
x' &= \alpha x + \beta y\\
y' &= \alpha y - \beta x
\end{aligned}$$

and the solution is

$$\begin{pmatrix} x\\ y\end{pmatrix} = e^{\alpha t}\begin{pmatrix} c_1\cos\beta t + c_2\sin\beta t\\ c_2\cos\beta t - c_1\sin\beta t\end{pmatrix}$$

$and\; x^2+y^2 = e^{2\alpha t}(c_1^2+c_2^2) \to \infty \;as\; t\to\infty$

Therefore, this is a spiral source.

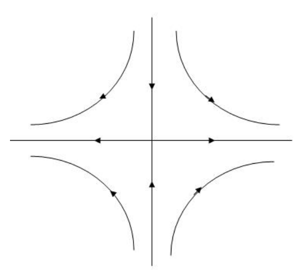


g) One negative and one positive eigenvalue: $-\mu < 0 < \lambda$

$$\begin{aligned}
x' &= \lambda x \qquad\qquad \frac{dy}{dx} = \frac{y'}{x'} = -\frac{\mu}{\lambda}\frac{y}{x} \to y = y_0 x_0^{\frac{\mu}{\lambda}}\frac{1}{x^{\frac{\mu}{\lambda}}}\\
y' &= -\mu y
\end{aligned}$$

Directions?

Quadrant 1) $y' < 0, x' > 0$



$(0,0)$ is stable if solutions start along the y-axis and unstable if they start anywhere else.
This is called a saddle.
For the nonlinear system, behaviour near (0,0) depends on initial conditions. Some solutions will approach (0,0), but most will move away. Therefore, (0,0) is unstable.

Exercise. Solve the case when $x' = -\lambda x, y' = \mu y$ and draw the phase portrait. Justify the behaviour on the axes.

In general, if the equation is hyperbolic, then

(1) if all eigenvalues have negative real part, the equilibrium is stable.

(2) If there is an eigenvalue with positive real part, the equilibrium is unstable. In particular,

1.i) if all eigenvalues are real and negative, the equilibrium is a sink.

1.ii) if all eigenvalues are complex with negative real part, it's a spiral sink.

2.i) if all eigenvalues are real and positive, it's a source.

2.ii) if eigenvalues are complex with positive real part, it's a spiral source.

2.iii) if some eigenvalues have positive real part and some have negative real part, it's a saddle.

4.3 Stability

An equilibrium is stable if nearby solutions stay nearby for all future time. Formally, if $\bar{x} \in \mathbb{R}^n$ is an equilibrium point of $x' = F(x)$ then $\bar{x}$ is stable if for every neighbourhood Q of $\bar{x}$ in $\mathbb{R}^n$, there is a neighbourhood Q_1 of $\bar{x}$ inside Q such that every solution $x(t)$ with $x(0) = x_0$ in Q_1 is defined and remains in Q for all $t > 0$. An equilibrium is asymptotically stable if nearby solutions actually converge to it as $t \to \infty$. Formally, if Q can be chosen so that $\bar{x}$ is stable, and $\lim_{t\to\infty} x(t) = \bar{x} \; \forall \; x(t) \in Q_1$, then $\bar{x}$ is asymptotically stable.

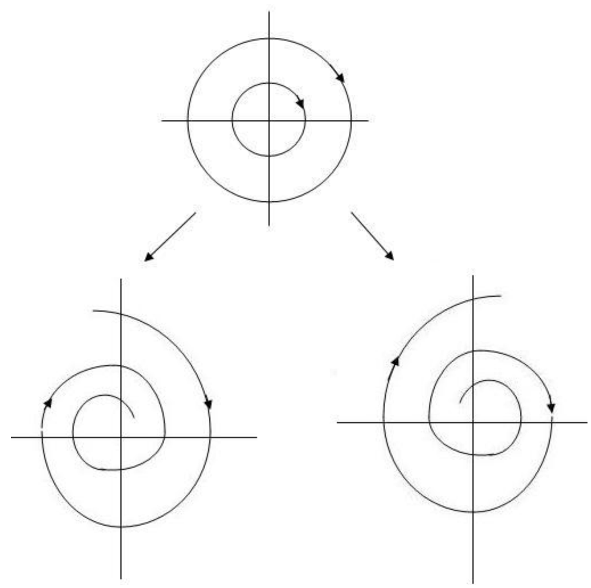
An equilibrium is unstable if it is not stable. Formally, there is a neighbourhood Q of $\bar{x}$ such that for every neighbourhood Q_1 of $\bar{x}$ in Q, there is at least one solution $x(t)$ starting at $x(0) \in Q$, that does not lie entirely in Q for all $t > 0$.

Example.

Sinks are asymptotically stable, sources and saddles are unstable. A center, for example,

$$x' = 2x$$
$$y' = -2y$$

is stable, but not asymptotically stable. As we've seen, the usefulness of this is limited, since a small nonlinear perturbation can make it into a sink or source.



4.4 Nonhomogeneous Linear Systems

For nonhomogeneous linear systems of the form

$$\vec{y}' = A\vec{y} + \vec{g}$$

solutions look like

$$\vec{y} = \vec{y}^{(h)} + \vec{y}^{(p)}$$

We already know how to solve homogeneous linear systems, so the problem comes down to finding particular solutions.

As before, we have two methods: undetermined coefficients and variation of parameters.

4.5 Method of Undetermined Coefficients for Systems

As before, this method only applies if the components of $\vec{g}$ are integer powers of t, exponential functions or sines or cosines.

Example. Solve

$$\begin{aligned} y_1' &= 2y_1 - 4y_2 + 2t^2 + 10t \\ y_2' &= y_1 - 3y_2 + t^2 + 9t + 3 \end{aligned}$$

We can write this in the form

$$\begin{aligned} \vec{y}\,' &= A\vec{y} + \vec{g} \\ &= \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \vec{y} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix} \end{aligned}$$

The homogeneous equation is

$$\vec{y}\,' = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \vec{y}$$

We need to find the eigenvalues and eigenvectors of A.

We have

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{bmatrix} \\ &= (2-\lambda)(-3-\lambda) + 4 \\ &= \lambda^2 + \lambda - 2 \\ &= (\lambda+2)(\lambda-1) = 0 \end{aligned}$$

Thus $\lambda = 1, -2$.

For $\lambda = 1$, we have

$$\begin{aligned} A\vec{m} &= \vec{m} \\ (A - 1I)\vec{m} &= 0 \\ \begin{bmatrix} 2-1 & -4 \\ 1 & -3-1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Thus $m_1 - 4m_2 = 0$, so eigenvectors are of the form

$$\vec{m} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} r \qquad \text{for } r \in \mathbb{R}, r \neq 0$$

For $\lambda = -2$, we have

$$A\vec{n} = -2\vec{n}$$
$$(A+2I)\vec{n} = 0$$
$$\begin{bmatrix} 2-(-2) & -4 \\ 1 & -3-(-2) \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus $n_1 - n_2 = 0$, so eigenvectors are of the form

$$\vec{n} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s \qquad \text{for } s \in \mathbb{R}, s \neq 0$$

Thus the solution to the homogeneous part is

$$\vec{y}^{(h)} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

Next we need to find a particular solution.

We assume $\vec{y}^{(p)}$ is in the form

$$\vec{y}^{(p)} = \vec{u} + \vec{v}t + \vec{w}t^2$$

where we need to determine $\vec{u}$, $\vec{v}$ and $\vec{w}$. Substituting into the original equation, we have

$$\vec{y}^{(p)\prime} = \vec{v} + 2\vec{w}t = A(\vec{u} + \vec{v}t + \vec{w}t^2) + \vec{g}$$

We thus have

$$\vec{v} + 2\vec{w}t = A\vec{u} + A\vec{v}t + A\vec{w}t^2 + \vec{g}$$
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 2w_1 t \\ 2w_2 t \end{bmatrix} = \begin{bmatrix} 2u_1 - 4u_2 \\ u_1 - 3u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 - 4v_2 \\ v_1 - 3v_2 \end{bmatrix} t + \begin{bmatrix} 2w_1 - 4w_2 \\ w_1 - 3w_2 \end{bmatrix} t^2 + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$$

We can equate like terms in the polynomial. The coefficients of t^2 are

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2w_1 - 4w_2 \\ w_1 - 3w_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$w_1 = 3w_2 - 1$$
$$2w_1 - 4w_2 + 2 = 0$$
$$2(3w_2 - 1) - 4w_2 + 2 = 0$$
$$w_2 = 0$$
$$w_1 = -1$$

From the coefficients of t, we have

$$\begin{bmatrix} 2w_1 \\ 2w_2 \end{bmatrix} = \begin{bmatrix} 2v_1 - 4v_2 \\ v_1 - 3v_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$
$$-2 = 2v_1 - 4v_2 + 10$$
$$v_1 = 2v_2 - 6$$
$$0 = v_1 - 3v_2 + 9$$
$$0 = (2v_2 - 6) - 3v_2 + 9$$
$$v_2 = 3$$
$$v_1 = 0$$

From the constant terms, we have

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2u_1 - 4u_2 \\ u_1 - 3u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$
$$\begin{aligned} 0 &= 2u_1 - 4u_2 \\ u_1 &= 2u_2 \\ 3 &= u_1 - 3u_2 + 3 \\ 3 &= 2u_2 - 3u_2 + 3 \\ u_2 &= 0 \\ u_1 &= 0 \end{aligned}$$

Thus

$$\begin{aligned} \vec{y}^{(p)} &= \vec{u} + \vec{v}t + \vec{w}t^2 \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} t^2 \end{aligned}$$

Hence, the complete solution is

$$\vec{y} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -t^2 \\ 3t \end{bmatrix}.$$

4.5.1 Modification term

If a term in g involves $e^{\lambda t}$ where λ is an eigenvalue of A, then we need a modification term. Instead of assuming a term of the form $\vec{u}e^{\lambda t}$, we thus assume a term of the form $\vec{u}te^{\lambda t} + \vec{v}e^{\lambda t}$. (This is slightly different to what we did earlier, where we only included the $\vec{u}te^{\lambda t}$ term, but the additional $\vec{v}e^{\lambda t}$ term is needed here.)

Example. Solve

$$\begin{aligned} y_1' &= 2y_1 - 4y_2 - 6e^{-2t} \\ y_2' &= y_1 - 3y_2 + 2e^{-2t} \end{aligned}$$

First we need the homogeneous solution.

The homogeneous solution is the same as the previous example, so

$$\vec{y}^{(h)} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

Next we try and guess the form for the particular solution.

Since e^{-2t} is already part of the homogeneous solution, we choose a particular solution of the form $\vec{y}^{(p)} = \vec{u}te^{-2t} + \vec{v}e^{-2t}$. Substituting, we have

$$\vec{y}^{(p)\prime} = \vec{u}e^{-2t} - 2\vec{u}te^{-2t} - 2\vec{v}e^{-2t} = A\vec{u}te^{-2t} + A\vec{v}e^{-2t} + \vec{g}$$

Equating the te^{-2t} terms on both sides, we have

$$-2\vec{u} = A\vec{u}$$

By definition, $\vec{u}$ is thus an eigenvector of A, corresponding to $\lambda = -2$. Hence

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s \qquad \text{for } s \in \mathbb{R}, s \neq 0$$

Equating the e^{-2t} terms (including those in g) gives

$$\begin{aligned} \vec{u} - 2\vec{v} &= A\vec{v} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ (A + 2I)\vec{v} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} s - \begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ \left(\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} s+6 \\ s-2 \end{bmatrix} \\ \begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} s+6 \\ s-2 \end{bmatrix} \end{aligned}$$

We thus have

$$\begin{aligned} 4v_1 - 4v_2 &= s + 6 \\ v_1 - v_2 &= s - 2 \end{aligned}$$

There are no solutions unless

$$\begin{aligned} s + 6 &= 4(s - 2) \\ 3s &= 14 \\ s &= \frac{14}{3} \end{aligned}$$

We thus have two variables and one equation:

$$\begin{aligned} v_1 - v_2 &= \frac{14}{3} - 2 \\ v_1 &= v_2 + \frac{8}{3} \end{aligned}$$

Hence one of the variables is free, say $v_2 = k$. Thus

$$\begin{aligned} \vec{v} &= \begin{bmatrix} k + \frac{8}{3} \\ k \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} k + \begin{bmatrix} \frac{8}{3} \\ 0 \end{bmatrix} \end{aligned}$$

(Note that there is no value of k that makes $\vec{v} = \vec{0}$, which is why we had to include $\vec{v}$.)

We can choose any value of k that we like (since this term is absorbed into $\vec{y}^{(h)}$), so let's choose $k = 0$ for simplicity.

Thus the particular solution is

$$\begin{aligned} \vec{y}^{(p)} &= \vec{u}te^{-2t} + \vec{v}e^{-2t} \\ &= \begin{bmatrix} \frac{14}{3} \\ \frac{14}{3} \end{bmatrix} te^{-2t} + \begin{bmatrix} \frac{8}{3} \\ 0 \end{bmatrix} e^{-2t} \end{aligned}$$

Hence the solution is

$$\begin{aligned}\vec{y} &= \vec{y}^{(h)} + \vec{y}^{(p)} \\ &= c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{14}{3} \\ \frac{14}{3} \end{bmatrix} te^{-2t} + \begin{bmatrix} \frac{8}{3} \\ 0 \end{bmatrix} e^{-2t}\end{aligned}$$

Note that we could have stuck to a general value for k. In that case,

$$\begin{aligned}\vec{y} &= c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{14}{3} \\ \frac{14}{3} \end{bmatrix} te^{-2t} + \begin{bmatrix} \frac{8}{3} \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} ke^{-2t} \\ &= c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + (c_2 + k) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{14}{3} \\ \frac{14}{3} \end{bmatrix} te^{-2t} + \begin{bmatrix} \frac{8}{3} \\ 0 \end{bmatrix} e^{-2t}\end{aligned}$$

But since both c_2 and k are arbitrary, this is identical to the form we found.

4.6 Variation of Parameters for Systems

The method of variation of parameters replaces the constants in the homogeneous solution with functions and then simplifies the problem by equating as many derivatives to zero as possible.

Consider the system

$$\vec{y}' = A(t)\vec{y} + \vec{g}(t)$$

The solution is $\vec{y} = \vec{y}^{(h)} + \vec{y}^{(p)}$, where $\vec{y}^{(h)}$ is the solution to the homogeneous system and $\vec{y}^{(p)}$ is the particular solution.

Let $Y(t)$ be the matrix with columns $\vec{y}^{(1)}, \vec{y}^{(2)}, \dots, \vec{y}^{(n)}$, where the $\vec{y}^{(j)}$'s satisfy

$$\begin{aligned}\vec{y}^{(h)} &= c_1\vec{y}^{(1)} + c_2\vec{y}^{(2)} + \cdots + c_n\vec{y}^{(n)} \\ &= \begin{bmatrix} c_1 y_1^{(1)} + c_2 y_1^{(2)} + \cdots + c_n y_1^{(n)} \\ c_1 y_2^{(1)} + c_2 y_2^{(2)} + \cdots + c_n y_2^{(n)} \\ \vdots \\ c_1 y_n^{(1)} + c_2 y_n^{(2)} + \cdots + c_n y_n^{(n)} \end{bmatrix} \\ &= \begin{bmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ \vdots & \vdots & & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= Y(t)\vec{c}\end{aligned}$$

For the particular solution, the method of variation of parameters replaces the constant vector $\vec{c}$ by a variable vector $\vec{u}(t)$. Thus

$$\vec{y}^{(p)} = Y(t)\vec{u}(t)$$

Substituting into the original equation, we have

$$Y'\vec{u} + Y\vec{u}' = AY\vec{u} + \vec{g}$$

Since $\vec{y}^{(h)}$ solves the homogeneous system, we have

$$\vec{y}^{(1)\prime} = A\vec{y}^{(1)}, \qquad \vec{y}^{(2)\prime} = A\vec{y}^{(2)}, \qquad \cdots \qquad \vec{y}^{(n)\prime} = A\vec{y}^{(n)}.$$

We can write these n vector equations as a single matrix equation $Y' = AY$. Note that Y is invertible, because the n solutions of a basis are linearly independent.

Thus

$$\begin{aligned} Y\vec{u}' &= \vec{g} \\ \vec{u}' &= Y^{-1}\vec{g} \\ u &= \int_{t_0}^{t} Y^{-1}(\tilde{t})\vec{g}(\tilde{t})d\tilde{t} + \vec{c} \end{aligned}$$

If $\vec{c} = \vec{0}$, then this gives us the particular solution. However, if we leave $\vec{c}$ general, then we get the general solution

$$\begin{aligned} \vec{y} &= Y\vec{u} \\ &= Y\vec{c} + Y\int_{t_0}^{t} Y^{-1}(\tilde{t})\vec{g}(\tilde{t})d\tilde{t} \end{aligned}$$

Example. Use variation of parameters to solve the previous example:

$$\begin{aligned} y_1' &= 2y_1 - 4y_2 - 6e^{-2t} \\ y_2' &= y_1 - 3y_2 + 2e^{-2t} \end{aligned}$$

We have

$$\begin{aligned} Y &= [\vec{y}^{(1)}\ \vec{y}^{(2)}] \\ &= \begin{bmatrix} 4e^t & e^{-2t} \\ e^t & e^{-2t} \end{bmatrix} \\ \vec{g} &= \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} \end{aligned}$$

Inverting Y, we have

$$\begin{aligned} Y^{-1} &= \frac{1}{4e^{-t} - e^{-t}} \begin{bmatrix} e^{-2t} & -e^{-2t} \\ -e^t & 4e^t \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{2t} & 4e^{2t} \end{bmatrix} \end{aligned}$$

Multiplying by $\vec{g}$, we get

$$\begin{aligned} Y^{-1}\vec{g} &= \frac{1}{3} \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{2t} & 4e^{2t} \end{bmatrix} \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -6e^{-3t} - 2e^{-3t} \\ 6 + 8 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -8e^{-3t} \\ 14 \end{bmatrix} \end{aligned}$$

Integrating and choosing the constant of integration to be zero, we have

$$\begin{aligned} \vec{u}(t) &= \int \frac{1}{3} \begin{bmatrix} -8e^{-3t} \\ 14 \end{bmatrix} dt \\ &= \begin{bmatrix} 8e^{-3t}/9 \\ 14t/3 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 8e^{-3t} \\ 42t \end{bmatrix} \end{aligned}$$

Finally, we have

$$\begin{aligned} Y\vec{u} &= \begin{bmatrix} 4e^t & e^{-2t} \\ e^t & e^{-2t} \end{bmatrix} \frac{1}{9} \begin{bmatrix} 8e^{-3t} \\ 42t \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 32e^{-2t} + 42te^{-2t} \\ 8e^{-2t} + 42te^{-2t} \end{bmatrix} \end{aligned}$$

The general solution is thus

$$\begin{aligned} Y\vec{u} &= c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{14}{3} \\ \frac{14}{3} \end{bmatrix} te^{-2t} + \frac{1}{9} \begin{bmatrix} 32 \\ 8 \end{bmatrix} e^{-2t} \\ &= c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{14}{3} \\ \frac{14}{3} \end{bmatrix} te^{-2t} + \frac{1}{9} \begin{bmatrix} 8 \\ 8 \end{bmatrix} e^{-2t} + \frac{1}{9} \begin{bmatrix} 24 \\ 0 \end{bmatrix} e^{-2t} \\ &= c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + \tilde{c}_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{14}{3} \\ \frac{14}{3} \end{bmatrix} te^{-2t} + \begin{bmatrix} \frac{8}{3} \\ 0 \end{bmatrix} e^{-2t} \end{aligned}$$

where we have absorbed the $\frac{1}{9} \begin{bmatrix} 8 \\ 8 \end{bmatrix} e^{-2t}$ term into the $c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$ term. This matches our previous answer.

6 Laplace Transforms

6.1 Introduction and definition

Thus far, our method of solving an initial value problem has been to find a general solution and then apply the initial conditions. This means we are solving a much bigger problem than we need to. The Laplace Transform will allow us to solve an initial problem without requiring the general solution first. It is a generalisation of the method of undetermined coefficients that works on linear differential equations with constant coefficients.

Definition 6.1. *The Laplace Transform assigns to the function $f(t)$ a new function, $F(s)$ defined by*

$$\boxed{F(s) = \int_0^\infty e^{-st} f(t)dt.}$$

$F(s)$ is called the Laplace transform of $f(t)$ and we write

$$F(s) = \mathcal{L}[f(t)]$$

Remarks:

1. The formula defines a function $F(s)$. That is, it assigns a number $F(s)$ to each numerical value of s. This means that s acts like a constant inside the integral sign.
2. Since the upper limit of integration is infinite, we have an improper integral. Recall from calculus that this is interpreted as a limit of proper integrals:

$$\int_0^\infty e^{-st} f(t)dt = \lim_{h\to\infty} \int_0^h e^{-st} f(t)dt$$

3. Although the integrand may seem complicated at first, the formulas for the transforms of the functions we most often deal with turn out to be quite reasonable. What's more, we only need to derive each of them once.
4. We haven't specified the domain of F. To do so, we'll restrict s if need be in order to ensure the result is finite.

Example. Calculate $\mathcal{L}[e^{\lambda t}]$.
From the definition, we have

$$\mathcal{L}[e^{\lambda t}] = \int_0^\infty e^{-st} e^{\lambda t} dt = \lim_{h\to\infty} \int_0^h e^{-(s-\lambda)t} dt.$$

The integral will vary depending on whether $s - \lambda = 0$ or not. If $s = \lambda$, then

$$\mathcal{L}[e^{\lambda t}] = \lim_{h\to\infty} \int_0^h 1dt = \lim_{h\to\infty} h = \infty$$

If $s \neq \lambda$, then

$$\mathcal{L}[e^{\lambda t}] = \lim_{h\to\infty} \int_0^h e^{-(s-\lambda)t} dt = \lim_{h\to\infty} \left[-\frac{1}{s-\lambda} e^{-(s-\lambda)h} + \frac{1}{s-\lambda} \right]$$

These limits only makes sense if $s > \lambda$. Thus the function $F(s)$ is

$$F(s) = \mathcal{L}[e^{\lambda t}] = \frac{1}{s-\lambda} \qquad \text{for } s > \lambda$$

In particular, if $\lambda = 0$, we have

$$\mathcal{L}[1] = \frac{1}{s} \qquad \text{for } s > 0$$

Example. Calculate $\mathcal{L}[\cos\beta t]$ for $\beta \neq 0$.
From the definition, we have

$$\mathcal{L}[\cos\beta t] = \int_0^\infty e^{-st}\cos\beta t dt = \lim_{h\to\infty}\int_0^h e^{-st}\cos\beta t dt.$$

Let $I = \int_0^h e^{-st}\cos\beta t dt$. Then we have

$$u = e^{-st} \qquad v' = \cos\beta t$$
$$u' = -se^{-st} \qquad v = \frac{1}{\beta}\sin\beta t$$

$$I = \frac{1}{\beta}e^{-st}\sin\beta t\Big|_0^h + \frac{s}{\beta}\int_0^h e^{-st}\sin\beta t dt$$

$$u = e^{-st} \qquad v' = \sin\beta t$$
$$u' = -se^{-st} \qquad v = -\frac{1}{\beta}\cos\beta t$$

$$\begin{aligned}
I &= \frac{1}{\beta}e^{-st}\sin\beta t\Big|_0^h + \frac{s}{\beta}\left[-\frac{1}{\beta}e^{-st}\cos\beta t - \frac{s}{\beta}\int_0^h e^{-st}\cos\beta t dt\right] \\
&= \left[\frac{1}{\beta}e^{-st}\sin\beta t - \frac{s}{\beta^2}e^{-st}\cos\beta t\right]_0^h - \frac{s^2}{\beta^2}I \\
I\left(1+\frac{s^2}{\beta^2}\right) &= \left[\frac{1}{\beta}e^{-sh}\sin\beta h - \frac{s}{\beta^2}e^{-sh}\cos\beta h\right] - \left[0 - \frac{s}{\beta^2}\right] \\
I &= \left(\frac{\beta^2}{\beta^2+s^2}\right)\left[\frac{1}{\beta}e^{-sh}\sin\beta h - \frac{s}{\beta^2}e^{-sh}\cos\beta h\right] + \left(\frac{\beta^2}{\beta^2+s^2}\right)\frac{s}{\beta^2}
\end{aligned}$$

Taking limits as $h \to \infty$, we have

$$\mathcal{L}[\cos\beta t] = \frac{s}{\beta^2+s^2} \qquad \text{for } s > 0.$$

(We impose the condition $s > 0$ to stop solutions going to infinity.)
Example. Find $\mathcal{L}[\sin\beta t]$ for $\beta \neq 0$.
We could do this by mimicking the previous example. But notice that the first integration by parts gave us

$$I = \frac{1}{\beta}e^{-st}\sin\beta t\Big|_0^h + \frac{s}{\beta}\int_0^h e^{-st}\sin\beta t dt$$

The integral on the left is what we want to find in this example. Thus

$$\int_0^h e^{-st}\sin\beta t dt = -\frac{\beta}{s}\left[\frac{1}{\beta}e^{-sh}\sin\beta h\right] + \frac{\beta}{s}I$$

Taking the limit as $h \to \infty$ with $s > 0$, we have

$$\mathcal{L}[\sin\beta t] = \frac{\beta}{s}\mathcal{L}[\cos\beta t] = \frac{\beta}{s}\frac{s}{\beta^2+s^2} = \frac{\beta}{\beta^2+s^2} \qquad \text{for } s > 0$$

Example. Calculate $\mathcal{L}[t^n]$ where n is a positive integer.

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st}t^n dt = \lim_{h\to\infty}\int_0^h e^{-st}t^n dt$$

Using integration by parts, we have

$$u = t^n \qquad\qquad v' = e^{-st}$$
$$u' = nt^{n-1} \qquad\qquad v = -\frac{1}{s}e^{-st}$$

$$\begin{aligned}\int_0^h e^{-st}t^n dt &= -\frac{t^n}{s}e^{-st}\Big|_0^h + \frac{n}{s}\int_0^h e^{-st}t^{n-1}dt \\ &= -\frac{h^n}{s}e^{-sh} + \frac{n}{s}\int_0^h e^{-st}t^{n-1}dt\end{aligned}$$

Taking the limit as $h \to \infty$, the term outside the integrand will go to zero if $s > 0$ (by L'Hôpital's rule). We thus have

$$\begin{aligned}\lim_{h\to\infty}\int_0^h e^{-st}t^n dt &= -\lim_{h\to\infty}\frac{h^n}{s}e^{-sh} + \lim_{h\to\infty}\frac{n}{s}\int_0^h e^{-st}t^{n-1}dt \\ \mathcal{L}[t^n] &= \frac{n}{s}\mathcal{L}[t^{n-1}] \qquad \text{for } s > 0\end{aligned}$$

We thus have a recursive formula, which can eventually bring us down to $\mathcal{L}[1] = \frac{1}{s}$. That is

$$\begin{aligned}\mathcal{L}[t^n] &= \frac{n}{s}\mathcal{L}[t^{n-1}] = \frac{n(n-1)}{s^2}\mathcal{L}[t^{n-2}] = \cdots = \frac{n!}{s^n}\mathcal{L}[1] \\ &= \frac{n!}{s^{n+1}} \qquad \text{for } s > 0\end{aligned}$$

Example. Calculate $\mathcal{L}[3e^{2t} - t^4]$.
By definition, we have

$$\mathcal{L}[3e^{2t} - t^4] = \int_0^\infty e^{-st}(3e^{2t} - t^4)dt$$

We could use integration by parts, but we don't need to, because we know that integrals are linear. Thus

$$\begin{aligned}\int_0^\infty e^{-st}(3e^{2t} - t^4)dt &= 3\int_0^\infty e^{-st}e^{2t}dt - \int_0^\infty e^{-st}t^4 dt \\ &= 3\mathcal{L}[e^{2t}] - \mathcal{L}[t^4] \\ &= 3\frac{1}{s-2} - \frac{4!}{s^5} \\ &= \frac{3}{s-2} - \frac{24}{s^5}\end{aligned}$$

Note that $\mathcal{L}[e^{2t}]$ is defined for $s > 2$ while $\mathcal{L}[t^4]$ is defined for $s > 0$. Thus the new formula will be valid for all values of s that are higher than both 0 and 2; ie for $s > 2$.

This gives us the idea for finding Laplace transforms of linear combinations of any known functions. Because it is an integral, the Laplace transform is linear. That is, for any two functions $f_1(t)$ and $f_2(t)$ and constants c_1 and c_2, we have

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1\mathcal{L}[f_1(t)] + c_2\mathcal{L}[f_2(t)]$$

Example. Find $\mathcal{L}[5 - e^{-7t} + 3\cos 4t]$.

By linearity,

$$\begin{aligned}\mathcal{L}[5-e^{-7t}+3\cos 4t] &= 5\mathcal{L}[1]-\mathcal{L}[e^{-7t}]+3\mathcal{L}[\cos 4t]\\ &= 5\frac{1}{s}-\frac{1}{s+7}+3\frac{s}{s^2+16}\\ &= \frac{5}{s}-\frac{1}{s+7}+\frac{3s}{s^2+16} \qquad \text{for } s>0\end{aligned}$$

Example. Find $\mathcal{L}[\cosh \beta t]$

We have

$$\begin{aligned}\mathcal{L}[\cosh \beta t] &= \lim_{h\to\infty}\int_0^h e^{-st}\cosh \beta t dt\\ &= \lim_{h\to\infty}\int_0^h e^{-st}\frac{e^{\beta t}+e^{-\beta t}}{2}dt\\ &= \frac{1}{2}\lim_{h\to\infty}\int_0^h \left(e^{(\beta-s)t}+e^{-(\beta+s)t}\right)dt\end{aligned}$$

If $s=\beta$, the integral is

$$\begin{aligned}\frac{1}{2}\lim_{h\to\infty}\int_0^h \left(1+e^{-2\beta t}\right)dt &= \frac{1}{2}\lim_{h\to\infty} h-\frac{e^{-2\beta h}}{2}+\frac{1}{2}\\ &= \infty\end{aligned}$$

which we don't want, so we can rule this case out. Similarly, if $s=-\beta$, the integral is

$$\begin{aligned}\frac{1}{2}\lim_{h\to\infty}\int_0^h \left(e^{2\beta t}+1\right)dt &= \frac{1}{2}\lim_{h\to\infty}\frac{e^{2\beta h}}{2}+h-\frac{1}{2}\\ &= \infty\end{aligned}$$

so we can also rule this out. If $s\neq\pm\beta$, the integral is

$$\begin{aligned}\frac{1}{2}\lim_{h\to\infty}\int_0^h \left(e^{(\beta-s)t}+e^{-(\beta+s)t}\right)dt &= \frac{1}{2}\lim_{h\to\infty}\left[\frac{e^{(\beta-s)t}}{\beta-s}-\frac{e^{-(\beta+s)t}}{\beta+s}\right]_0^h\\ &= \frac{1}{2}\lim_{h\to\infty}\left[\left(\frac{e^{(\beta-s)h}}{\beta-s}-\frac{e^{-(\beta+s)h}}{\beta+s}\right)-\left(\frac{1}{\beta-s}-\frac{1}{\beta+s}\right)\right]\end{aligned}$$

For the integral to converge, we need $\beta-s<0$ and $\beta+s>0$. This means that $s>\beta$ and $s>-\beta$. Thus if $s>\beta$, we have

$$\begin{aligned}\frac{1}{2}\left[0-0-\frac{1}{\beta-s}+\frac{1}{\beta+s}\right] &= \frac{1}{2}\cdot\frac{-(\beta+s)+\beta-s}{(\beta+s)(\beta-s)}\\ &= \frac{1}{2}\cdot\frac{-2s}{\beta^2-s^2}\\ &= \frac{s}{s^2-\beta^2}\end{aligned}$$

Exercise. Show that

$$\mathcal{L}[\sinh \beta t]=\frac{\beta}{s^2-\beta^2}$$

The Laplace transform turns out to be very useful for functions that may be defined in pieces.

Example. Find $L[u_a(t)]$ where $a>0$ and

$$u_a(t)=\begin{cases}0 & \text{if } 0\le t<a\\ 1 & \text{if } a\le t.\end{cases}$$

By definition

$$\begin{aligned}
\mathcal{L}[u_a(t)] &= \int_0^\infty e^{-st} u_a(t)dt \\
&= \int_0^a e^{-st} u_a(t)dt + \int_a^\infty e^{-st} u_a(t)dt \\
&= \int_0^a e^{-st} 0dt + \int_a^\infty e^{-st} 1dt \\
&= \lim_{h\to\infty} \int_a^h e^{-st} dt \\
&= \lim_{h\to\infty} \begin{cases} h - a & \text{if } s = 0 \\ -\frac{e^{-sh}}{s} + \frac{e^{-sa}}{s} & \text{if } s \neq 0 \end{cases} \\
&= \frac{e^{-sa}}{s} \qquad \text{for } s > 0
\end{aligned}$$

The final step in being able to apply the Laplace transform to initial-value problems will require us to retrieve a function from its Laplace transform.

Definition 6.2. *If $F(s) = \mathcal{L}[f(t)]$, then we say that $f(t)$ is an inverse Laplace transform of $F(s)$ and write*

$$f(t) = \mathcal{L}^{-1}[F(s)].$$

The relationship between $\mathcal{L}$ and $\mathcal{L}^{-1}$ is like the relationship between differentiation and integration. Just as a table of integrals starts from a "backward" reading of differentiation formulas, so an inverse transform table begins with transform formulas read backwards.

We can thus summarise our results thus far.

Laplace transform	Inverse transform
$\mathcal{L}\left[e^{\lambda t}\right] = \frac{1}{s-\lambda}$	$\mathcal{L}^{-1}\left[\frac{1}{s-\lambda}\right] = e^{\lambda t}$
$\mathcal{L}[1] = \frac{1}{s}$	$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$
$\mathcal{L}[\cos\beta t] = \frac{s}{s^2+\beta^2}$	$\mathcal{L}^{-1}\left[\frac{s}{s^2+\beta^2}\right] = \cos\beta t$
$\mathcal{L}[\sin\beta t] = \frac{\beta}{s^2+\beta^2}$	$\mathcal{L}^{-1}\left[\frac{\beta}{s^2+\beta^2}\right] = \sin\beta t$
$\mathcal{L}\left[\frac{t^{n-1}}{(n-1)!}\right] = \frac{1}{s^n}$	$\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$
$\mathcal{L}[\cosh\beta t] = \frac{s}{s^2-\beta^2}$	$\mathcal{L}^{-1}\left[\frac{s}{s^2-\beta^2}\right] = \cosh\beta t$
$\mathcal{L}[\sinh\beta t] = \frac{\beta}{s^2-\beta^2}$	$\mathcal{L}^{-1}\left[\frac{\beta}{s^2-\beta^2}\right] = \sinh\beta t$

Recall that $\mathcal{L}$ is linear. Thus if $\mathcal{L}[f_1(t)] = F_1(s)$ and $\mathcal{L}[f_2(t)] = F_2(s)$, then $\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$. Read backwards, this gives

$$\begin{aligned}
\mathcal{L}^{-1}[c_1 F_1(t) + c_2 F_2(t)] &= c_1 f_1(s) + c_2 f_2(s) \\
&= c_1 \mathcal{L}^{-1}[F_1(s)] + c_2 \mathcal{L}^{-1}[F_2(s)]
\end{aligned}$$

Thus $\mathcal{L}^{-1}$ is linear. Hence if we know the inverse transforms of some basic functions, we can find inverse transforms of their linear combinations.

Example. Find $\mathcal{L}^{-1}\left[\frac{2}{s+3} - \frac{6}{s^2+25} + \frac{1}{s}^7\right]$.

Using linearity, we have

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2}{s+3}-\frac{6}{s^2+25}+\frac{1}{s^7}\right] &= \mathcal{L}^{-1}\left[\frac{2}{s+3}\right]-\mathcal{L}^{-1}\left[\frac{6}{s^2+25}\right]+\mathcal{L}^{-1}\left[\frac{1}{s^7}\right]\\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]-\frac{6}{5}\mathcal{L}^{-1}\left[\frac{5}{s^2+5^2}\right]+\frac{1}{6!}\mathcal{L}^{-1}\left[\frac{6!}{s^7}\right]\\ &= 2e^{-3t}-\frac{6}{5}\sin 5t+\frac{1}{720}t^6\end{aligned}$$

6.2 Initial value problems

We want to solve an ODE with constant coefficients

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = g(t).$$

The idea is for the Laplace transform to change a differential equation into an algebraic equation when the transform is applied to both sides. Since $\mathcal{L}$ is linear, we have

$$a_n\mathcal{L}[x^{(n)}] + a_{n-1}\mathcal{L}[x^{(n-1)}] + \cdots + a_1\mathcal{L}[x'] + a_0\mathcal{L}[x] = \mathcal{L}[g(t)].$$

We know how to find $\mathcal{L}[g(t)]$ for a number of choices of $g(t)$. However, we need a way to deal with the terms $\mathcal{L}[x^{(k)}]$.

Let's start by considering $\mathcal{L}[x'(t)]$. By definition,

$$\mathcal{L}[x'(t)] = \int_0^\infty e^{-st}x'(t)dt = \lim_{h\to\infty}\int_0^h e^{-st}x'(t)dt$$

We use integration by parts:

$$\begin{aligned} u &= e^{-st} & v' &= x'(t)\\ u' &= -se^{-st} & v &= x(t)\end{aligned}$$

$$\begin{aligned}\mathcal{L}[x'(t)] &= \lim_{h\to\infty}\left(e^{-st}x(t)\Big|_0^h + s\int_0^h e^{-st}x(t)dt\right)\\ &= \lim_{h\to\infty} e^{-sh}x(h) - x(0) + s\mathcal{L}[x(t)]\end{aligned}$$

For the functions we are interested in,

$$\lim_{h\to\infty} e^{-sh}x(h) = 0$$

as long as s is sufficiently large. We thus have the following:

First Differentiation Formula (k=1): $\mathcal{L}[x'(t)] = s\mathcal{L}[x] - x(0)$.

Example. Solve the initial value problem $x' = t$, $x(0) = 2$ using a Laplace transform.
Applying $\mathcal{L}$ to both sides of the IDE, we have

$$\begin{aligned}\mathcal{L}[x'(t)] &= \mathcal{L}[t]\\ s\mathcal{L}[x] - x(0) &= \frac{1}{s^2}\\ s\mathcal{L}[x] - 2 &= \frac{1}{s^2} && \text{using the initial condition}\\ \mathcal{L}[x] &= \frac{1}{s^3}+\frac{2}{s}\\ x &= \mathcal{L}^{-1}\left[\frac{1}{s^3}+\frac{2}{s}\right] && \text{taking the inverse transform}\\ &= \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]+2\mathcal{L}^{-1}\left[\frac{1}{s}\right] && \text{since } \mathcal{L}^{-1} \text{ is linear}\\ &= \frac{1}{2}t^2+2 && \text{from the table}\end{aligned}$$

How to deal with higher order derivatives?
For $k = 2$, we note that $x''(t) = (x'(t))'$. Thus

$$\begin{aligned}
\mathcal{L}[x''(t)] &= \mathcal{L}[(x'(t))'] \\
&= s\mathcal{L}[x'(t)] - x'(0) && \text{applying the first differentiation formula} \\
&= s\Big(s\mathcal{L}[x(t)] - x(0)\Big) - x'(0) \\
&= s^2\mathcal{L}[x] - sx(0) - x'(0)
\end{aligned}$$

Repeated application of this process yields the following general formula.

First Differentiation Formula: $\mathcal{L}[x^{(k)}] = s^k\mathcal{L}[x] - s^{k-1}x(0) - s^{k-2}x'(0) - \cdots - sx^{(k-2)}(0) - x^{(k-1)}(0)$

Example. Solve the initial value problem

$$x''' - x'' = 0, \qquad x(0) = x'(0) = x''(0) = 3$$

Applying $\mathcal{L}$ to both sides of the ODE, we get

$$\mathcal{L}[x'''] - \mathcal{L}[x''] = \mathcal{L}[0] = 0$$

By the differentiation formula,

$$\begin{aligned}
\mathcal{L}[x'''] &= s^3\mathcal{L}[x] - s^2x(0) - sx'(0) - x''(0) \\
&= s^3\mathcal{L}[x] - 3s^2 - 3s - 3 \\
\mathcal{L}[x''] &= s^2\mathcal{L}[x] - sx(0) - x'(0) \\
&= s^2\mathcal{L}[x] - 3s - 3
\end{aligned}$$

We thus have

$$\begin{aligned}
\mathcal{L}[x'''] - \mathcal{L}[x''] &= 0 \\
\Big(s^3\mathcal{L}[x] - 3s^2 - 3s - 3\Big) - \Big(s^2\mathcal{L}[x] - 3s - 3\Big) &= 0 \\
(s^3 - s^2)\mathcal{L}[x] - 3s^2 &= 0 \\
\mathcal{L}[x] &= \frac{3s^2}{s^3 - s^2} \\
&= \frac{3}{s-1}
\end{aligned}$$

Taking the inverse transform, we have

$$\begin{aligned}
x &= \mathcal{L}^{-1}\left[\frac{3}{s-1}\right] \\
&= 3\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] \\
&= 3e^t
\end{aligned}$$

This example illustrates the three steps in solving an initial value problem by Laplace transforms:

1. Transform the ODE, incorporating initial data, by means of the first differentiation formula
2. Solve algebraically for $\mathcal{L}[x]$ in terms of s
3. Obtain x as the inverse transform of $\mathcal{L}[x]$

In these examples, the third step was unusually easy, because we recognised the inverse transform from the table. In most cases, we need to rewrite $\mathcal{L}[x]$ in order to recognise its inverse transform. This is done by means of partial fraction decomposition of quotients of polynomials.

6.2.1 Partial Fraction Decomposition

Each polynomial $q(s)$ with real coefficients can, at least in theory, be factored as a number (the leading coefficient) times a product of irreducible polynomials of two kinds: **linear factors** $s-a$, where a is a real root of the polynomial; and **irredicible quadratic factors** s^2+bs+c (with $b^2<4c$), corresponding to pairs of complex roots. If $p(s)$ is a polynomial whose degree is strictly less than the degree of $q(s)$, then the rational expression $p(s)/q(s)$ can be written as a sum according to the following rules:

i. If $(s-a)^m$ is the highest power of $s-a$ that divides $q(s)$, then the sum should include terms of the form

$$\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+\cdots+\frac{A_m}{(s-a)^m}$$

ii. If $(s^2+bs+c)^m$ is the highest power of the irreducible quadratic s^2+bs+c that divides $q(s)$, then the sum should include terms of the form

$$\frac{B_1s+C_1}{s^2+bs+c}+\frac{B_2s+C_2}{(s^2+bs+c)^2}+\cdots+\frac{B_ms+C_m}{(s^2+bs+c)^m}$$

In general, we obtain the partial fraction decomposition of a rational expression $p(s)/q(s)$, once we know how to factor $q(s)$, by first using long division to rewrite the original quotient as a polynomial plus a new quotient whose numerator has degree less than the denominator $q(s)$. We then write this new quotient as a sum of terms of the form (i) and (ii), corresponding to all the factors of $q(s)$.

Example. Solve the initial value problem

$$x'-x=2\sin t, \qquad x(0)=0$$

First transform both sides of the ODE.

$$\begin{aligned}\mathcal{L}[x']-\mathcal{L}[x]&=\mathcal{L}[2\sin t]\\(s\mathcal{L}[x]-x(0))-\mathcal{L}[x]&=\frac{2}{s^2+1}\\(s-1)\mathcal{L}[x]&=\frac{2}{s^2+1}\end{aligned}$$

Next, solve for $\mathcal{L}[x]$.

$$\begin{aligned}\mathcal{L}[x]&=\frac{2}{(s-1)(s^2+1)}\\x&=\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+1)}\right]\end{aligned}$$

To find the inverse transform, we look for a partial fraction decomposition of $\mathcal{L}[x]$.
This is of the form

$$\begin{aligned}\frac{2}{(s-1)(s^2+1)}&=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}\\2&=A(s^2+1)+(Bs+C)(s-1)\end{aligned}$$

Since A, B and C are constants, they must hold true for any values of s. We can thus choose s judiciously.

$$\begin{aligned} s = 1: \qquad\qquad\qquad\qquad & 2 = A(2) + 0 \\ & A = 1 \\ s = 0: \qquad\qquad\qquad\qquad & 2 = A + (0 + C)(-1) \\ & 2 = 1 - C \\ & C = -1 \end{aligned}$$

The choices $s = 1$ and $s = 0$ are "smart" choices, because they eliminate unknown constants. The final choice of s has no obvious "smart" choice, but any s will do, since we know the other constants. Thus

$$\begin{aligned} s = -1: \qquad\qquad\qquad\qquad & 2 = A(2) + (-B + C)(-2) \\ & 2 = 2 + 2B + 2 \\ & B = -1 \end{aligned}$$

Thus the partial fraction decomposition is

$$\begin{aligned} \frac{2}{(s-1)(s^2+1)} &= \frac{A}{s-1} + \frac{Bs + C}{s^2+1} \\ &= \frac{1}{s-1} + \frac{-s-1}{s^2+1} \end{aligned}$$

We can thus take the inverse transform in order to find the solution.

$$\begin{aligned} x &= \mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+1)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-1} + \frac{-s-1}{s^2+1}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\ &= e^t - \cos t - \sin t \end{aligned}$$

Example. Solve the intial value problem

$$x'' - x = 0, \qquad\qquad x(0) = 3, x'(0) = 1$$

First transform both sides of the ODE.

$$\begin{aligned} \mathcal{L}[x''] - \mathcal{L}[x] &= 0 \\ \left(s^2\mathcal{L}[x] - sx(0) - x'(0)\right) - \mathcal{L}[x] &= 0 \\ \left(s^2\mathcal{L}[x] - 3s - 1\right) - \mathcal{L}[x] &= 0 \\ (s^2 - 1)\mathcal{L}[x] &= 3s + 1 \\ \mathcal{L}[x] &= \frac{3s+1}{s^2-1} \end{aligned}$$

Next, we find the partial fraction decomposition of $\mathcal{L}[x]$.

$$
\begin{aligned}
\frac{3s+1}{s^2-1} &= \frac{3s+1}{(s+1)(s-1)} \\
&= \frac{A}{s+1} + \frac{B}{s-1} \\
3s+1 &= A(s-1) + B(s+1)
\end{aligned}
$$

Thus

$$
\begin{aligned}
s = 1: \qquad & 4 = B(2) \\
& B = 2 \\
s = -1: \qquad & -2 = A(-2) \\
& A = 1
\end{aligned}
$$

We can thus use the inverse transform to solve for x.

$$
\begin{aligned}
x &= \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{2}{s-1}\right] \\
&= e^{-t} + 2e^{t}
\end{aligned}
$$

Example. Solve the initial value problem

$$
x'' - 2x' = 4, \qquad x(0) = -1, x'(0) = 2
$$

$$
\begin{aligned}
\mathcal{L}[x''] - 2\mathcal{L}[x'] &= \mathcal{L}[4] \\
\left(s^2\mathcal{L}[x] - sx(0) - x'(0)\right) - 2\left(s\mathcal{L}[x] - x(0)\right) &= \frac{4}{s} \\
\left(s^2\mathcal{L}[x] + s - 2\right) - 2\left(s\mathcal{L}[x] + 1\right) &= \frac{4}{s} \\
(s^2 - 2s)\mathcal{L}[x] &= \frac{4}{s} + 4 - s \\
(s^2 - 2s)\mathcal{L}[x] &= \frac{4 + 4s - s^2}{s} \\
\mathcal{L}[x] &= \frac{4 + 4s - s^2}{s^2(s-2)}
\end{aligned}
$$

Using partial fractions, we have

$$
\begin{aligned}
\frac{4 + 4s - s^2}{s^2(s-2)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} \\
4 + 4s - s^2 &= As(s-2) + B(s-2) + Cs^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
s = 0: \qquad & 4 = 0 + B(-2) + 0 \\
& B = -2 \\
s = 2: \qquad & 8 = 0 + 0 + 4C \\
& C = 2 \\
s = 1 \qquad & 7 = A(-1) + B(-1) + C \\
& 7 = -A + 2 + 2 \\
& A = -3
\end{aligned}
$$

The solution is thus

$$
\begin{aligned}
x &= \mathcal{L}^{-1}\left[\frac{4+4s-s^2}{s^2(s-2)}\right] \\
&= \mathcal{L}^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{-3}{s}+\frac{-2}{s^2}+\frac{2}{s-2}\right] \\
&= -3\mathcal{L}^{-1}\left[\frac{1}{s}\right]-2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]+2\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] \\
&= -3-2t+2e^{2t}
\end{aligned}
$$

6.2.2 Laplace transforms of integrals

Note that the function

$$g(t)=\int_0^t f(t)dt$$

satisfies $g'(t)=f(t)$ and $g(0)=0$. Taking Laplace transforms, we get

$$
\begin{aligned}
s\mathcal{L}[g(t)]-g(0) &= \mathcal{L}[f(t)] \\
\mathcal{L}[g(t)] &= \frac{1}{s}\mathcal{L}[f(t)]
\end{aligned}
$$

Setting $F(s)=\mathcal{L}[f(t)]$, we see that

$$\mathcal{L}\left[\int_0^t f(t)dt\right]=\frac{1}{s}F(s)$$

Alternately,

$$\mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t \mathcal{L}^{-1}[F(s)]dt$$

Example. Find $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+\beta^2)}\right]$.

We have

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s(s^2+\beta^2)}\right] &= \int_0^t \mathcal{L}^{-1}\left[\frac{1}{s^2+\beta^2}\right]dt \\
&= \frac{1}{\beta}\int_0^t \sin\beta t dt \\
&= -\frac{1}{\beta^2}\cos\beta t+\frac{1}{\beta^2}
\end{aligned}
$$

Example. Find $\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right]$.

We can deal with the $\frac{1}{s^2}$ term by integrating twice:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right] &= \int_0^t \int_0^t \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] dtdt \\
&= \int_0^t \int_0^t e^{2t} dtdt \\
&= \int_0^t \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) dt \\
&= \left[\frac{e^{2t}}{4} - \frac{t}{2}\right]_0^t \\
&= \left[\left(\frac{e^{2t}}{4} - \frac{t}{2}\right) - \left(\frac{1}{4} - 0\right)\right] \\
&= \frac{e^{2t}}{4} - \frac{t}{2} - \frac{1}{4}
\end{aligned}$$

6.3 First shift formula

6.3.1 t-shifting

Suppose we know $F(s) = \mathcal{L}[f(t)]$. Then the Laplace transform of $e^{\alpha t} f(t)$ is

$$\begin{aligned}
\mathcal{L}[e^{\alpha t} f(t)] &= \int_0^\infty e^{-st} e^{\alpha t} f(t) dt \\
&= \int_0^\infty e^{-(s-\alpha)t} f(t) dt \\
&= F(s-\alpha)
\end{aligned}$$

That is, multiplying a function by an exponential is equivalent to a translation in the Laplace transform.

First Shift Formula (t-shifting): If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[e^{\alpha t} f(t)] = F(s-\alpha)$

Example. Find $\mathcal{L}[t^3 e^{2t}]$.
We know that

$$\mathcal{L}[t^3] = F(s) = \frac{6}{s^4}.$$

Thus, by the first shift formula, $\mathcal{L}[t^3 e^{2t}]$ is obtained from $\mathcal{L}[t^3]$ by replacing s with $s-2$. Hence

$$\mathcal{L}[t^3 e^{2t}] = F(s-2) = \frac{6}{(s-2)^4}$$

Example. Find $\mathcal{L}[e^{-t} \cos 3t]$.
We know

$$\mathcal{L}[\cos 3t] = F(s) = \frac{s}{s^2+9}$$

Thus, replacing s with $s+1$, we have

$$\mathcal{L}[e^{-t} \cos 3t] = F(s+1) = \frac{s+1}{(s+1)^2+9}$$

Of course, every transform can be rewritten as an inverse transform formula. We can thus rewrite the first shift formula as

$$\mathcal{L}^{-1}[F(s-\alpha)] = e^{\alpha t} \mathcal{L}^{-1}[F(s)].$$

If we replace s with $s+\alpha$, then we obtain a formula that is easier to work with.

First shift formula (inverse version): $\mathcal{L}^{-1}[F(s)] = e^{\alpha t}\mathcal{L}^{-1}[F(s+\alpha)]$

Example. Find $\mathcal{L}^{-1}\left[\dfrac{3}{(s-2)^5}\right]$

We know how to deal with inverse transform powers of s, so substituting $s+2$ for s would turn our problem into one of this type. Thus

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{3}{(s-2)^5}\right] &= e^{2t}\mathcal{L}^{-1}\left[\frac{3}{(s+2-2)^5}\right]\\ &= e^{2t}\mathcal{L}^{-1}\left[\frac{3}{s^5}\right]\\ &= \frac{3}{4!}e^{2t}t^4\\ &= \frac{1}{8}e^{2t}t^4\end{aligned}$$

Example. Find $\mathcal{L}^{-1}\left[\dfrac{s}{(s-1)^2+4}\right]$.

If the denominator had the form s^2+4, we could handle this using trigonometric functions. Therefore we try the substitution of $s+1$ for s, which changes $(s-1)^2$ into s^2.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{(s-1)^2+4}\right] &= e^{t}\mathcal{L}^{-1}\left[\frac{s+1}{(s+1-1)^2+4}\right]\\ &= e^{t}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}+\frac{1}{s^2+4}\right]\\ &= e^{t}\cos 2t + \frac{1}{2}e^{t}\sin 2t\end{aligned}$$

In general, to deal with terms of the form $\dfrac{Bs+C}{s^2+bs+c}$, we first check to see whether the denominator can be factored. If it can, we use partial fractions; if it can't, we complete the square and shift.

Example. Solve the initial value problem

$$x'' + 2x' + 2x = 25te^t, \qquad x(0) = x'(0) = 0$$

Transforming both sides of the ODE, we have

$$\begin{aligned}s^2\mathcal{L}[x] + 2s\mathcal{L}[x] + 2\mathcal{L}[x] &= \frac{25}{(s-1)^2}\\ \mathcal{L}[x] &= \frac{25}{(s-1)^2(s^2+2s+2)}\\ &= \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{Cs+D}{s^2+2s+2}\end{aligned}$$

Thus using partial fractions, we have

$$
\begin{aligned}
& & 25 &= A(s-1)(s^2+2s+2) + B(s^2+2s+2) + (Cs+D)(s-1)^2 \\
& s=1 & 25 &= 5B \\
& & B &= 5 \\
& s=0 & 25 &= A(-1)(2) + 5(2) + D(1) \\
& & D &= 2A + 15 \\
& s=-1 & 25 &= A(-2)(1) + 5(1) + (-C+D)(4) \\
& & 20 &= -2A - 4C + 4D \\
& & &= -2A - 4C + 4(2A+15) \\
& & -40 &= 6A - 4C \\
& & C &= \frac{3}{2}A + 10
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& s=2 & 25 &= A(1)(10) + 5(10) + (2C+D)(1) \\
& & -25 &= 10A + 2C + D \\
& & -25 &= 10A + 2\left(\frac{3}{2}A + 10\right) + (2A+15) \\
& & -25 &= 10A + 3A + 20 + 2A + 15 \\
& & -60 &= 15A \\
& & A &= -4 \\
& & C &= \frac{3}{2}(-4) + 10 = 4 \\
& & D &= 2(-4) + 15 = 7
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathcal{L}[x] &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+2s+2} \\
&= \frac{-4}{s-1} + \frac{5}{(s-1)^2} + \frac{4s+7}{s^2+2s+2} \\
x &= \mathcal{L}^{-1}\left[\frac{-4}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{5}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{4s+7}{s^2+2s+2}\right] \\
&= -4\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 5\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{4s+7}{s^2+2s+2}\right] \\
&= -4e^t + 5te^t + \mathcal{L}^{-1}\left[\frac{4s+7}{s^2+2s+2}\right]
\end{aligned}
$$

using t-shifting for the first two terms. The third term requires us to complete the square in the denominator.

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{4s+7}{s^2+2s+2}\right] &= \mathcal{L}^{-1}\left[\frac{4s+7}{(s+1)^2+1}\right] \\
&= e^{-t}\mathcal{L}^{-1}\left[\frac{4(s-1)+7}{s^2+1}\right] && \text{by } t\text{-shifting} \\
&= e^{-t}\mathcal{L}^{-1}\left[\frac{4s+3}{s^2+1}\right] \\
&= 4e^{-t}\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + 3e^{-t}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\
&= 4e^{-t}\cos t + 3e^{-t}\sin t
\end{aligned}
$$

Thus the solution of the initial value problem is

$$x = -4e^t + 5te^t + 4e^{-t}\cos t + 3e^{-t}\sin t$$

6.3.2 Second differentiation formula

Another important formula comes from differentiating the Laplace transform. By definition,

$$\frac{d}{ds}\mathcal{L}[f(t)] = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt$$

For functions we are interested in (piecewise continuous and of exponential order), the differentiation can be carried out inside the integral sign. Thus

$$\begin{aligned}\frac{d}{ds}\mathcal{L}[f(t)] &= \int_0^\infty \left[\frac{\partial}{\partial s}e^{-st}f(t)\right]dt \\ &= -\int_0^\infty e^{-st}tf(t)dt \\ &= -\mathcal{L}[tf(t)] \\ \text{Conversely, } \mathcal{L}[tf(t)] &= -\frac{d}{ds}\mathcal{L}[f(t)]\end{aligned}$$

Repeated application of this formula gives a more general version.

Second differentiation formula: $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[f(t)]$

Example. Find $\mathcal{L}[te^{2t}\cos 3t]$

First we use the second differentiation formula to find $\mathcal{L}[t\cos 3t]$:

$$\begin{aligned}\mathcal{L}[t\cos 3t] &= -\frac{d}{ds}\mathcal{L}[\cos 3t] \\ &= -\frac{d}{ds}\left(\frac{s}{s^2+9}\right) \\ &= -\frac{s^2+9-s(2s)}{(s^2+9)^2} \\ &= \frac{s^2-9}{(s^2+9)^2}\end{aligned}$$

We now use the first shift formula to get

$$\begin{aligned}\mathcal{L}[te^{2t}\cos 3t] &= \frac{(s-2)^2-9}{((s-2)^2+9)^2} \\ &= \frac{s^2-4s-5}{(s^2-4s+13)^2}\end{aligned}$$

6.3.3 Integral of the transform

Let $F(s) = \mathcal{L}[f(t)]$. Then

$$\begin{aligned}
\int_s^\infty F(x)dx &= \int_s^\infty \int_0^\infty e^{-sx} f(x) dx ds \\
&= \int_0^\infty \int_s^\infty e^{-sx} f(x) ds dx \\
&= \int_0^\infty \lim_{h\to\infty} \int_s^h e^{-sx} f(x) ds dx \\
&= \int_0^\infty \lim_{h\to\infty} \left[\frac{e^{-sx} f(x)}{-x}\right]_s^h dx \\
&= \int_0^\infty \lim_{h\to\infty} \left[\frac{e^{-hx} f(x)}{-x} + \frac{e^{-sx} f(x)}{x}\right] dx \\
&= \int_0^\infty \frac{e^{-sx} f(x)}{x} dx \\
&= \mathcal{L}\left[\frac{f(t)}{t}\right]
\end{aligned}$$

Example. Find $\mathcal{L}\left[\frac{\sinh 3t}{t}\right]$.

Note that the Laplace transform of $\sinh 3t$ is only valid for $s > 3$. We have

$$\mathcal{L}\left[\frac{\sinh 3t}{t}\right] = \int_s^\infty \frac{3}{x^2 - 9} dx$$

Using partial fractions, we have

$$\begin{aligned}
\frac{3}{(x+3)(x-3)} &= \frac{A}{x+3} + \frac{B}{x-3} \\
3 &= A(x-3) + B(x+3) \\
x = 3: \qquad 3 &= B(6) \qquad\qquad B = \frac{1}{2} \\
x = -3: \qquad 3 &= A(-6) \qquad\qquad A = -\frac{1}{2} \\
\frac{3}{(x+3)(x-3)} &= \frac{-1/2}{x+3} + \frac{1/2}{x-3}
\end{aligned}$$

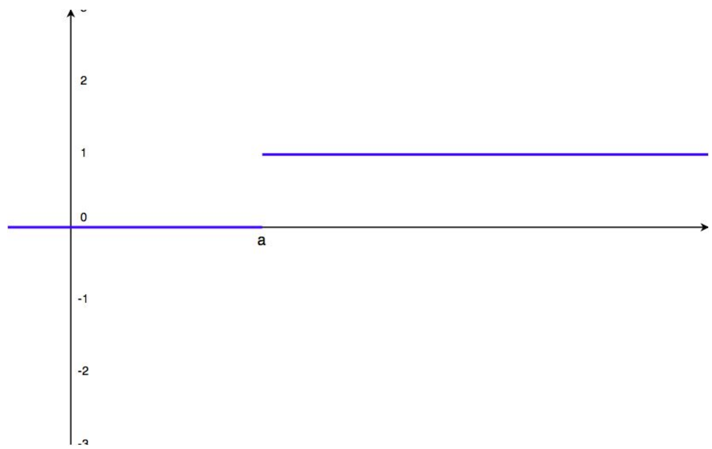
Thus

$$\begin{aligned}
\mathcal{L}\left[\frac{\sinh 3t}{t}\right] &= \lim_{h\to\infty} \frac{1}{2} \int_s^h -\frac{1}{x+3} + \frac{1}{x-3} dx \\
&= \lim_{h\to\infty} \frac{1}{2} \Big[-\ln(x+3) + \ln(x-3) \Big]_s^h \qquad \text{(this is well-defined since } x > 3) \\
&= \lim_{h\to\infty} \frac{1}{2} \left[-\ln\frac{x+3}{x-3} \right]_s^h \\
&= \lim_{h\to\infty} \frac{1}{2} \left[-\ln\frac{h+3}{h-3} + \ln\frac{s+3}{s-3} \right] \\
&= \frac{1}{2} \ln\frac{s+3}{s-3}
\end{aligned}$$

6.4 Piecewise continuous functions

Many phenomena are best defined by functions in pieces: circuits with switches, doses of drugs, sudden shocks. Our first task is to develop a better notation for functions defined in pieces. This is accomplished through the use of the unit step function:

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$



We already found that $\mathcal{L}[u_a(t)] = \frac{e^{-sa}}{s}$.

The unit step function has the effect of a mathematical "on" switch at $t = a$. If we multiply a function $f(t)$ by $u_a(t)$, then product will be zero until $t = 1$ and will switch to $f(t)$ thereafter:

$$u_a(t)f(t) = \begin{cases} 0 & \text{if } t < a \\ f(t) & \text{if } t \geq a \end{cases}$$

Thus, for example, the function

$$g(t) = \begin{cases} 0 & \text{if } t < 2 \\ e^{-t} & \text{if } t \geq 2 \end{cases}$$

can be rewritten as

$$g(t) = u_2(t)e^{-t}$$

Of course, we are often interested in functions that switch from one nonzero formula to another, such as

$$g(t) = \begin{cases} t^2 & \text{if } t < 3 \\ e^{-t} & \text{if } t \geq 3 \end{cases}$$

We can think of this as a function that starts out equal to t^2. At time $t = 3$, a switch does two things: it turns off the first formula and turns on the second. To turn off the formula $g(t) = t^2$ at $t = 3$, we subtract $u_3(t)t^2$. To turn on the formula $g(t) = e^{-t}$ at $t = 3$, we add $u_3(t)e^{-t}$. Thus

$$g(t) = t^2 + u_3(t)(-t^2 + e^{-t})$$

Example. Rewrite $g(t) = |2t - 1|$ in step-function notation.

$$g(t) = \begin{cases} -(2t-1) & \text{if } 2t-1 < 0 \\ 2t-1 & \text{if } 2t-1 \geq 0 \end{cases}$$

$$= \begin{cases} 1-2t & \text{if } t < \frac{1}{2} \\ 2t-1 & \text{if } t \geq \frac{1}{2} \end{cases}$$

The initial formula is $g(t) = 1 - 2t$. At time $t = \frac{1}{2}$, we use $u_{1/2}(t)$ to switch off $1 - 2t$ and to switch on $2t - 1$:

$$\begin{aligned} g(t) &= 1 - 2t + u_{1/2}(t)\left[-(1-2t) + (2t-1)\right] \\ &= 1 - 2t + u_{1/2}(t)[4t - 2] \end{aligned}$$

Example. Rewrite

$$g(t) = \begin{cases} 2 & \text{if } t < 1 \\ 3t & \text{if } 1 \le t < 2 \\ 5 & \text{if } t \ge 2 \end{cases}$$

in step-function notation.

The initial formula is the constant 2 and there are two switching times: $t = 1$ and $t = 2$. At $t = 1$, we use $u_1(t)$ to switch off $g(t) = 2$ and switch on $g(t) = 3t$. At $t = 2$, we use $u_2(t)$ to switch off the previous formula $g(t) = 3t$ and switch on $g(t) = 5$. Thus

$$g(t) = 2 + u_1(t)(-2 + 3t) + u_2(t)(-3t + 5)$$

Step-function notation lets us rewrite any function defined in pieces as a sum of terms of the form $u_a(t)f(t)$ and reduces the calculation of the transform of such a function to finding the transform of this new kind of term.

Don't forget that $\mathcal{L}$ is an integral over the domain $t \ge 0$. Thus any formula that affects only negative values of t has no effect on the transform. Hence we only consider transforming terms of the form $u_a(t)f(t)$ with $a \ge 0$.

The definition of $\mathcal{L}$ gives

$$\begin{aligned} \mathcal{L}[u_a(t)f(t)] &= \int_0^\infty e^{-st}u_a(t)f(t)dt \\ &= \int_0^a e^{-st}(0)f(t)dt + \int_a^\infty e^{-st}(1)f(t)dt \\ &= \int_a^\infty e^{-st}f(t)dt \end{aligned}$$

This differs from the Laplace transform of $f(t)$ in that the lower limit of integration is not zero. However, if we set $\tau = t - a$, then $d\tau = dt$, $\tau = 0$ when $t = a$ and $\tau \to \infty$ as $t \to \infty$. Hence, by rewriting our integral in terms of τ, we have

$$\begin{aligned} \mathcal{L}[u_a(t)f(t)] &= \int_{\tau=0}^{\tau=\infty} e^{-s(\tau+a)}f(\tau + a)d\tau \\ &= e^{-as}\int_0^\infty e^{-s\tau}f(\tau + a)d\tau \end{aligned}$$

The last integral is equal to the transform of $f(t + a)$. We thus have the following:

Second shift formula (s-shifting): For $a \ge 0, \mathcal{L}[u_a(t)f(t)] = e^{-as}\mathcal{L}[f(t + a)]$

Example. Find $\mathcal{L}[u_2(t)e^{-t}]$.

Using s-shifting, we have

$$\begin{aligned} \mathcal{L}[u_2(t)e^{-t}] &= e^{-2s}\mathcal{L}[e^{-(t+2)}] \\ &= e^{-2s}e^{-2}\mathcal{L}[e^{-t}] && \text{(since the Laplace transform is linear)} \\ &= e^{-2(s+1)}\frac{1}{s+1} \end{aligned}$$

Example. Find $\mathcal{L}[|2t - 1|]$.

We saw earlier that

$$|2t - 1| = 1 - 2t + u_{1/2}(t)(4t - 2)$$

Thus

$$\begin{aligned}\mathcal{L}[|2t-1|] &= \mathcal{L}[1] - 2\mathcal{L}[t] + e^{-s/2}\mathcal{L}\left[4\left(t+\frac{1}{2}\right)-2\right] \\ &= \frac{1}{s} - \frac{2}{s^2} + e^{-s/2}\mathcal{L}[4t] \\ &= \frac{1}{s} - \frac{2}{s^2} + \frac{4e^{-s/2}}{s^2}\end{aligned}$$

Exercise. Find $\mathcal{L}[g(t)]$, where

$$g(t) = \begin{cases} 2 & \text{if } t < 1 \\ 3t & \text{if } 1 \le t < 2 \\ 5 & \text{if } t \ge 2 \end{cases}$$

If we set $f(t-a) = h(t)$, then the second shift formula says

$$\begin{aligned}\mathcal{L}[u_a(t)f(t-a)] &= \mathcal{L}[u_a(t)h(t)] \\ &= e^{-as}\mathcal{L}[h(t+a)] \\ &= e^{-as}\mathcal{L}[f(t)]\end{aligned}$$

We thus have the following:

Second shift formula (inverse version): If $\mathcal{L}^{-1}[F(s)] = f(t)$, then $\mathcal{L}^{-1}[e^{-as}F(s)] = u_a(t)f(t-a)$

Example. Find $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+4}\right]$.

We first find

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] \\ &= \frac{1}{2}\sin 2t\end{aligned}$$

Then the inverse transform version of the second shift formula says

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+4}\right] &= u_3(t)f(t-3) \\ &= u_3(t)\frac{1}{2}\sin(2(t-3)) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 3 \\ \frac{1}{2}\sin(2t-6) & \text{if } t \ge 3 \end{cases}\end{aligned}$$

Example. Solve the initial value problem

$$x'' - x = \begin{cases} t & \text{if } t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$$

with initial conditions $x(0) = x'(0) = 0$.

We first rewrite the ODE using $u_1(t)$.

$$x'' - x = t + u_1(t)(-t)$$

Next we transform both sides and solve for $\mathcal{L}[x]$.

$$s^2\mathcal{L}[x] - sx(0) - x'(0) - \mathcal{L}[x] = \frac{1}{s^2} + e^{-s}\mathcal{L}[-(t+1)]$$
$$(s^2-1)\mathcal{L}[x] = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$
$$\mathcal{L}[x] = \frac{1}{s^2(s^2-1)} - e^{-s}\left(\frac{1}{s^2(s^2-1)} + \frac{1}{s(s^2-1)}\right)$$

We need to use partial fractions.

$$\frac{1}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-1}$$
$$1 = As(s+1)(s-1) + B(s+1)(s-1) + Cs^2(s-1) + Ds^2(s+1)$$
$$s=0 \qquad 1 = B(-1) \qquad B=-1$$
$$s=1 \qquad 1 = D(2) \qquad D=\frac{1}{2}$$
$$s=-1 \qquad 1 = C(-2) \qquad C=-\frac{1}{2}$$
$$s=2 \qquad 1 = A(2)(3)(1) + B(3)(1) + C(4)(1) + D(4)(3)$$
$$1 = 6A - 3 - 2 + 6 \qquad A=0$$
$$\frac{1}{s^2(s+1)(s-1)} = -\frac{1}{s^2} - \frac{1/2}{s+1} + \frac{1/2}{s-1}$$

To find the second part, we use partial fractions again.

$$\frac{1}{s(s+1)(s-1)} = \frac{E}{s} + \frac{F}{s+1} + \frac{G}{s-1}$$
$$1 = E(s+1)(s-1) + Fs(s-1) + Gs(s+1)$$
$$s=0 \qquad 1 = E(-1) \qquad E=-1$$
$$s=1 \qquad 1 = G(2) \qquad G=\frac{1}{2}$$
$$s=-1 \qquad 1 = F(-1)(-2) \qquad F=\frac{1}{2}$$
$$\frac{1}{s(s+1)(s-1)} = -\frac{1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1}$$

Hence

$$\frac{1}{s^2(s^2-1)} + \frac{1}{s(s^2-1)} = -\frac{1}{s^2} - \frac{1/2}{s+1} + \frac{1/2}{s-1} - \frac{1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1}$$
$$= -\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}$$

We thus use the inverse transform to solve for x.
That is,

$$x = \mathcal{L}^{-1}\left[-\frac{1}{s^2} - \frac{1/2}{s+1} + \frac{1/2}{s-1}\right] - \mathcal{L}^{-1}\left[e^{-s}\left(-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}\right)\right]$$
$$= -t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - \mathcal{L}^{-1}\left[e^{-s}\left(-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}\right)\right]$$

We need to calculate

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}\right] \\ &= -t - 1 + e^t \end{aligned}$$

Using the inverse version of the second shift formula, we have

$$\begin{aligned} \mathcal{L}^{-1}\left[e^{-s}\left(-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}\right)\right] &= u_1(t)f(t-1) \\ &= u_1(t)\left[-(t-1) - 1 + e^{t-1}\right] \\ &= u_1(t)\left[-t + e^{t-1}\right] \end{aligned}$$

Hence the solution is

$$\begin{aligned} x &= -t + \frac{1}{2}e^t - \frac{1}{2}e^{-t} - u_1(t)\left[-t + e^{t-1}\right] \\ &= \begin{cases} -t + \frac{1}{2}e^t - \frac{1}{2}e^{-t} & \text{if } t < 1 \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} - e^{t-1} & \text{if } t \geq 1 \end{cases} \end{aligned}$$

Exercise. Check that both x and x' are continuous at 1.

6.4.1 Dirac delta function

The Dirac delta function $\delta(t)$ is characterised by the following two properties:

$$(1) \qquad \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

and

$$(2) \qquad \int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

for any function $f(t)$ that is continuous on an open interval containing $t = 0$.

By shifting the argument of $\delta(t)$, we have $\delta(t-a) = 0$ for $t \neq a$ and

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

for any function $f(t)$ that is continuous on an interval containing $t = a$.

Setting $f(t) = e^{-st}$, we have, for $a \geq 0$,

$$\int_0^{\infty} e^{-st}\delta(t-a)dt = \int_{-\infty}^{\infty} e^{-st}\delta(t-a)dt = e^{-as}$$

Thus, for $a \geq 0$,

$$\mathcal{L}[\delta(t-a)] = e^{-as}$$

Example. Solve the initial value problem

$$x'' + 9x = 3\delta(t-\pi), \qquad x(0) = 1, x'(0) = 0$$

Taking the Laplace transform of the ODE, we have

$$\begin{aligned} s^2\mathcal{L}[x] - sx(0) - x'(0) + 9\mathcal{L}[x] &= 3e^{-\pi s} \\ (s^2+9)\mathcal{L}[x] - s &= 3e^{-\pi s} \\ \mathcal{L}[x] &= \frac{s}{s^2+9} + \frac{3e^{-\pi s}}{s^2+9} \end{aligned}$$

Using the translation property, we thus have

$$\begin{aligned} x &= \cos 3t + [\sin 3(t-\pi)]u_\pi(t) \\ &= \begin{cases} \cos 3t & t < \pi \\ \cos 3t + \sin 3(t-\pi) & t \geq \pi \end{cases} \end{aligned}$$

6.5 Convolution

Definition 6.3. *Given two functions $f(t)$ and $g(t)$, we define a new function, called the convolution of f and g, denoted $f * g$, by the rule*

$$(f*g)(t) = \int_0^t f(t-u)g(u)du$$

The formula assigns a numerical value, $(f*g)(t)$, to each specific value of t so that, as far as integration is concerned, t acts like a constant.

The limits of integration refer to u: we integrate from $u = 0$ to $u = t$.

In practice, we try to rewrite $f(t-u)$ in terms of functions of t and u alone before integrating.

Example. Find $(f*g)(t)$ when $f(t) = e^{2t}$ and $g(t) = e^{3t}$.

We have

$$\begin{aligned} (f*g)(t) &= \int_0^t f(t-u)g(u)du \\ &= \int_0^t e^{2(t-u)}e^{3u}du \\ &= e^{2t}\int_0^t e^u du \\ &= e^{2t}(e^t - 1) \\ &= e^{3t} - e^{2t} \end{aligned}$$

Convolution shares many properties with multiplication:

- the distributive law: $f*(c_1g_1 + c_2g_2) = c_1(f*g_1) + c_2(f*g_2)$
- the associative law: $f*(g*h) = (f*g)*h$
- the commutative law: $f*g = g*f$

However, not every property carries over. For example, the convolution of $g(t)$ with the constant $f(t) = 1$ is not $g(t)$:

$$(1*g)(t) = \int_0^t g(u)du \neq g(t)$$

In Laplace transforms, convolution turns into products.

If $f(t)$ and $g(t)$ both have Laplace transforms, then $\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$

This also applies to inverse transforms.

If $F(s)$ and $G(s)$ both have inverse Laplace transforms, then $\mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[F(s)] * \mathcal{L}^{-1}[G(s)]$

Example. Find $\mathcal{L}^{-1}\left[\dfrac{1}{(s-2)(s-3)}\right]$.
We could do this by partial fractions, of course. Alternatively, we can use convolution:

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{(s-2)(s-3)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{(s-2)}\right] * \mathcal{L}^{-1}\left[\frac{1}{(s-3)}\right]\\ &= e^{2t} * e^{3t}\\ &= e^{3t} - e^{2t} \qquad \text{(from the previous example)}\end{aligned}$$

Example. Find $\mathcal{L}^{-1}\left[\dfrac{s}{(s^2+1)^2}\right]$.

We can't use partial fractions here, since the function is already written in its partial fraction decomposition. However, we can think of this as the product of $\frac{s}{(s^2+1)}$ and $\frac{1}{(s^2+1)}$. Then

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] &= \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right]\\ &= \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]\\ &= \cos t * \sin t\\ &= \int_0^t \cos(t-u)\sin u du\end{aligned}$$

We now use trigonometric identities to simplify this integral

$$\begin{aligned}\int_0^t \cos(t-u)\sin u du &= \int_0^t (\cos t\cos u + \sin t \sin u)\sin u du\\ &= \cos t\int_0^t \cos u \sin u du + \sin t\int_0^t \sin^2 u du\\ &= \cos t\int_0^t \cos u\sin u du + \frac{1}{2}\sin t\int_0^t (1-\cos 2u)du\end{aligned}$$

$$\begin{aligned}w &= \sin u\\ \frac{dw}{du} &= \cos u\\ du &= \frac{dw}{\cos u}\end{aligned}$$

Thus the integral is

$$\begin{aligned}\int_0^t \cos(t-u)\sin u du &= \cos t\int_{u=0}^{u=t} w\cos u\frac{dw}{\cos u} + \frac{1}{2}\sin t\int_0^t (1-\cos 2u)du\\ &= \cos t\left.\frac{w^2}{2}\right|_{u=0}^{u=t} + \frac{1}{2}\sin t\left[u - \frac{\sin 2u}{2}\right]_{u=0}^{u=t}\\ &= \cos t\left.\frac{\sin^2 u}{2}\right|_{u=0}^{u=t} + \frac{1}{2}\sin t\left[u-\frac{\sin 2u}{2}\right]_{u=0}^{u=t}\\ &= \frac{1}{2}\cos t\sin^2 t + \frac{1}{2}\sin t\left[t - \frac{\sin 2t}{2}\right]\end{aligned}$$

Exercise. Find $\mathcal{L}^{-1}\left[\dfrac{1}{(s^2+1)^2}\right]$.

Example. Solve the initial value problem

$$x'' + x = \cos t \qquad\qquad x(0) = x'(0) = 0$$

We transform both sides of the ODE:

$$\begin{aligned}
s^2\mathcal{L}[x] - sx(0) - x'(0) + \mathcal{L}[x] &= \mathcal{L}[\cos t] \\
(s^2+1)\mathcal{L}[x] &= \frac{s}{s^2+1} \\
\mathcal{L}[x] &= \frac{s}{(s^2+1)^2} \\
x &= \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] \\
&= \frac{1}{2}\cos t \sin^2 t + \frac{1}{2}\sin t\left[t - \frac{\sin 2t}{2}\right] \qquad \text{(from the previous example)}
\end{aligned}$$

Example. Solve the initial value problem

$$x^{iv} - x = \begin{cases} 2 & \text{if } t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$$

with initial conditions $x(0) = x'(0) = x''(0) = 0$, $x'''(0) = 2$.

In step-function notation, the ODE reads

$$x^{iv} - x = 2 - 2u_3(t)$$

so its transform is

$$\begin{aligned}
s^4\mathcal{L}[x] - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0) - \mathcal{L}[x] &= \frac{2}{s} - \frac{2}{s}e^{-3s} \\
(s^4-2)\mathcal{L}[x] &= 2 + \frac{2}{s} - \frac{2}{s}e^{-3s}
\end{aligned}$$

Solving for $\mathcal{L}[x]$, we have

$$\mathcal{L}[x] = \frac{2}{s^4-2} + \frac{2}{s(s^4-2)} - \frac{2}{s(s^4-2)}e^{-3s}$$

Using partial fractions on the first term, we have

$$\frac{2}{s^4-2} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$2 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+1)$$

$s = 1$ $\qquad 2 = A(2)(2) \qquad A = \frac{1}{2}$

$s = -1$ $\qquad 2 = B(-2)(2) \qquad B = -\frac{1}{2}$

$s = 0$ $\qquad 2 = A - B + D(-1)$

$\qquad 2 = \frac{1}{2} + \frac{1}{2} - D \qquad D = -1$

$s = 2$ $\qquad 2 = A(3)(5) + B(5) + (2C+D)(3)$

$\qquad 2 = \frac{15}{2} - \frac{5}{2} + 6C - 3 \qquad C = 0$

$$\frac{2}{s^4-2} = \frac{1/2}{s-1} - \frac{1/2}{s+1} - \frac{1}{s^2+1}$$

Thus the inverse transform is

$$\mathcal{L}^{-1}\left[\frac{2}{s^4-2}\right] = \frac{1}{2}e^t - \frac{1}{2}e^{-t} - \sin t$$

The second term could also be decomposed into partial fractions. However, note that it is the same as the first term, multiplied by $\frac{1}{s}$. We can thus use convolution to obtain its inverse transform from the previous one.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2}{s(s^4-2)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s}\right] * \mathcal{L}^{-1}\left[\frac{2}{s^4-1}\right] \\
&= 1 * \left(\frac{1}{2}e^t - \frac{1}{2}e^{-t} - \sin t\right) \\
&= \int_0^t \left(\frac{1}{2}e^u - \frac{1}{2}e^{-u} - \sin u\right) du \\
&= \frac{1}{2}(e^t - 1) + \frac{1}{2}(e^{-t} - 1) + (\cos t - 1) \\
&= \frac{1}{2}e^t - \frac{1}{2}e^{-t} + \cos t - 2
\end{aligned}$$

Finally, the third term is an exponential times the second. We obtain is inverse transform from that of the second term by means of the second shift formula.

$$\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s(s^4-1)}\right] = u_3(t)\left(\frac{1}{2}e^{t-3} + \frac{1}{2}e^{-(t-3)} + \cos(t-3) - 2\right)$$

Combining all three terms, we have

$$x = e^t + \cos t - \sin t - 2 - u_3(t)\left(\frac{1}{2}e^{t-3} + \frac{1}{2}e^{-(t-3)} + \cos(t-3) - 2\right)$$

5 Series solutions

5.1 Review

5.1.1 Infinite series

The expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

is called an infinite series.

5.1.2 Partial sum

Denote by s_n the partial sum

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$$

The partial sum is a finite series.

5.1.3 Convergence

If the partial sums converge to some real number s

$$\lim_{n\to\infty} s_n = s$$

then the series is convergent and the sum is said to be s. Otherwise, the series is said to be divergent. The series converges if it adds up to anything finite. If not (i.e. if it adds up to infinity or does not add up to a finite number, for example, it may oscillate) then it diverges.

5.1.4 Absolute convergence

If the series

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent then the series

$$\sum_{n=1}^{\infty} a_n$$

is said to be absolutely convergent. If a series is absolutely convergent, it converges.

5.1.5 Conditional convergence

If the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent but not absolutely convergent, then it is said to be conditionally convergent.
For example,

$$\sum \frac{(-1)^n}{n}$$

is conditionally convergent, while

$$\sum \frac{1}{n^2}$$

is absolutely convergent.

5.1.6 Ratio test

One can use the ratio test to test for convergence. Find

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

If L is less than 1 the series converges.
If L is greater than 1 then the series diverges.
If $L = 1$ we have no info ($L = 1$ in both examples above and for $\sum \frac{1}{n}$ which diverges.)

5.1.7 Power series

A power series has the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

More generally, we can have a power series about a constant a:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

A power series involves powers of x. The c_n's can be anything.

Example. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Applying the ratio test we get

$$a_n = \frac{x^n}{n^2} \qquad\qquad a_{n+1} = \frac{x^{n+1}}{(n+1)^2}$$

and so

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$$

We conclude that the power series is convergent for $|x| < 1$.

5.1.8 Taylor series

If a function $f(x)$ has a power series expansion about a it must have the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}$$

where $f^{(n)}(a)$ is the nth derivative of f at a.

Example. Find the Taylor series of e^x about zero and find its radius of convergence.

Since $f(x) = e^x$, all the derivatives are also e^x. The Taylor series is then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Applying the ratio test we get

$$a_n = \frac{x^n}{n!}$$

and so

$$\left|\frac{a_n+1}{a_n}\right| = \frac{|x|}{n+1}$$

For any given x the ratio goes to zero as n gets large. Thus the series converges for all x.
What about the Taylor series of e^x about 3?

$$\begin{aligned} f(x) &= e^x & f(3) &= e^3 \\ f'(x) &= e^x & f'(3) &= e^3 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= e^x & f^{(n)}(3) &= e^3 \end{aligned}$$

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n \\ &= \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n \end{aligned}$$

Note that most (i.e. differentiable) functions have a power series, but for a given power series it is unlikely we could find a corresponding function.

Example. Find the Taylor series and interval of convergence of $f(x) = \frac{4}{3x-2}$ around $x=0$.

5.1.9 Method 1

$$\begin{aligned} f(x) &= \frac{4}{3x-2} & f(0) &= \frac{4}{-2} \\ f'(x) &= \frac{-4\cdot 3}{(3x-2)^2} & f'(0) &= \frac{-4\cdot 3}{2^2} \\ f''(x) &= \frac{4\cdot 3^2\cdot 2}{(3x-2)^3} & f''(0) &= \frac{4\cdot 3^2\cdot 2}{-2^3} \\ f'''(x) &= \frac{-4\cdot 3^3\cdot 2\cdot 3}{(3x-2)^4} & f'''(0) &= \frac{-4\cdot 3^3\cdot 2\cdot 3}{2^4} \\ f^{iv}(x) &= \frac{4\cdot 3^4\cdot 2\cdot 3\cdot 4}{(3x-2)^5} & f^{iv}(0) &= \frac{4\cdot 3^4\cdot 2\cdot 3\cdot 4}{-2^5} \\ f^n(x) &= \frac{(-1)^n 4\cdot 3^n\cdot n!}{(3x-2)^{n+1}} & f^n(0) &= -\frac{4\cdot 3^n\cdot n!}{2^{n+1}} \end{aligned}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{4\cdot 3^n\cdot \cancel{n!}}{-2^{n+1}}\frac{x^n}{\cancel{n!}} = \sum_{n=0}^{\infty} \frac{4\cdot 3^n}{-2^{n+1}}x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{4}\cancel{3^{n+1}}}{\cancel{-}\cancel{2^{n+2}}}x^{\cancel{n+1}}\cdot\frac{\cancel{-}\cancel{2^{n+1}}}{\cancel{4}\cancel{3^n}\cancel{x^n}}\right| = \frac{3}{2}|x| < 1$$

$$|x| < \frac{2}{3}$$

Thus, the radius of convergence is $\frac{2}{3}$, so $-\frac{2}{3} < x < \frac{2}{3}$.
If $x = \frac{2}{3}$:

$$\sum_{n=0}^{\infty} \frac{4\cdot \cancel{3^n}}{-\cancel{2^{n+1}}}\frac{\cancel{2^n}}{\cancel{3^n}} = \sum_{n=0}^{\infty} -2 = -\infty$$

If $x = -\frac{2}{3}$:

$$\sum_{n=0}^{\infty} \frac{4\cdot \cancel{3^n}}{-2\cancel{^{n+1}}}\frac{(-1)^n\cancel{2^n}}{\cancel{3^n}} = \sum_{n=0}^{\infty} 2(-1)^{n+1}$$

This sum diverges. Therefore,

$$-\frac{2}{3} < x < \frac{2}{3}$$

5.1.10 Method 2

$$\begin{aligned}\frac{1}{1-y} &= 1+y+y^2+y^3+\ldots = \sum_{n=0}^{\infty} y^n \qquad |y|<1\\ \frac{4}{3x-2} &= \frac{-4}{2-3x}\\ &= -\frac{4}{2}\frac{1}{1-\frac{3x}{2}}\\ &= -\frac{4}{2}\sum_{n=0}^{\infty}\left(\frac{3x}{2}\right)^n\\ &= -\frac{4}{2}\sum_{n=0}^{\infty}\frac{3^n x^n}{2^n}\\ &= \sum_{n=0}^{\infty}\frac{-4\cdot 3^n\cdot x^n}{2^{n+1}}\end{aligned}$$

We know that $|\frac{3x}{2}| < 1$ so $|x| < \frac{2}{3}$ and we test endpoints individually as before.

Exercise. Find the Taylor series about 0 for $\sin x$ and $\cos x$.

5.2 Series solutions

5.2.1 Problem

We try to find solutions to

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

as infinite series. We do this because thus far we have dealt almost exclusively with the case of constant coefficients. In general it is hard or impossible to find "nice" solutions to non-constant coefficient problems.

5.2.2 Ordinary point

$x = x_0$ is an ordinary point if $P(x_0) \neq 0$.
i.e. we can divide through by $P(x_0)$ so

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

is well defined at (and around) $x = x_0$.

5.2.3 Singular point

$x = x_0$ is a singular point if $P(x_0) = 0$.
A singular point means a degenerate second order problem at $x = x_0$.

5.2.4 Regular singular point

$x = x_0$ is a regular singular point if it satisfies the following conditions

$$\begin{aligned}&i)\ P(x_0) = 0\\ &ii)\ (x-x_0)\frac{Q(x)}{P(x)} \text{ and } (x-x_0)^2\frac{R(x)}{P(x)} \text{ have convergent Taylor series about } x_0.\end{aligned}$$

If P, Q and R are polynomials then the conditions for a regular singular point are

$$\begin{aligned}&i)\ \lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \text{ is finite.}\\ &ii)\ \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} \text{ is finite.}\end{aligned}$$

Specifically, $(x-x_0)\frac{Q(x)}{P(x)}$ and $(x-x_0)^2\frac{R(x)}{P(x)}$ are defined at $x = x_0$ (and infinitely differentiable).

5.3 Euler-Cauchy equation

5.3.1 Equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

Example. Show that $x = 0$ is a regular singular point of $x^2y'' + \alpha xy' + \beta y = 0$.
We have

$$P(x) = x^2 \qquad Q(x) = \alpha x \qquad R(x) = \beta$$
$$P(0) = 0 \quad \text{is a singular point}$$
$$\frac{Q(x)}{P(x)} = \frac{\alpha}{x} \qquad \therefore x\frac{Q(x)}{P(x)} = \alpha = \alpha + 0x + 0x^2 + ... \text{ is the Taylor series}$$
$$\frac{R(x)}{P(x)} = \frac{\beta}{x^2} \qquad \therefore x^2\frac{R(x)}{P(x)} = \beta = \beta + 0x + 0x^2 + ... \text{is the Taylor series}$$

5.4 Worked Example

5.4.1 Problem

Try to find a series solution of

$$y'' - 2ty' - 2y = 0$$

5.4.2 Proposed solution

Try

$$y = \sum_{n=0}^{\infty} a_n t^n$$

If you prefer, use the longhand notation

$$y = a_0 + a_1 t + a_2 t^2 + ...$$

5.4.3 Find derivatives

$$y' = \sum_{n=0}^{\infty} n a_n t^{n-1} = a_1 + 2a_2 t + 3a_3 t^2 + ...$$
$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} = 2a_2 + 3 \cdot 2a_3 t + 4 \cdot 3a_4 t^2 + ...$$

Note that y' and y'' do not really have $n = 0$ or $n = 0, 1$ terms respectively. However, what we have here is still correct, since the coefficients are zero.

5.4.4 Substitution

We substitute back into the original equation to get

$$\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - 2t \sum_{n=0}^{\infty} n a_n t^{n-1} - 2 \sum_{n=0}^{\infty} a_n t^n = 0$$

which gives

$$\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - 2 \sum_{n=0}^{\infty} n a_n t^n - 2 \sum_{n=0}^{\infty} a_n t^n = 0$$

5.4.5 Rewriting the summation

We need to rewrite the summation in a form so that exponents match up. Note that if we decrease the n below the sum sign we can increase every n inside the summation by the same amount. In particular,

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}t^n$$

Now note that for $n = -2$ and $n = -1$ the contribution is zero

$$-2 + 2 = -1 + 1 = 0$$

and so

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}t^n = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$$

5.4.6 Group like powers

We then have the condition

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 2\sum_{n=0}^{\infty} na_n t^n - 2\sum_{n=0}^{\infty} a_n t^n = 0$$

and grouping like powers gives

$$(n+2)(n+1)a_{n+2} - 2na_n - 2a_n = 0$$

Alternatively, $y'' - 2ty' - 2y = 0$ becomes

$$2a_2 + 3\cdot 2a_3 t + 4\cdot 3a_4 t^2 + 5\cdot 4a_5 t^3 + ... - 2a_1 t - 2\cdot 2a_2 t^2 - 2\cdot 3a_3 t^3 - ... - 2a_0 - 2a_1 t - 2a_2 t^2 - 2a_3 t^3 - ... = 0$$

$$\begin{aligned} a_2 = a_0, \qquad & 3\cdot 2a_3 - 2a_1 - 2a_1 = 0 \\ & 4\cdot 3a_4 - 2\cdot 2a_2 - 2a_2 = 0 \\ & \quad \vdots \\ & (n+2)(n+1)a_{n+2} - 2na_n - 2a_n = 0 \end{aligned}$$

Whichever way you prefer, you still need to get this general formula.

5.4.7 Recurrence relation

We can now get a recurrence relation

$$a_{n+2} = \frac{2(n+1)a_n}{(n+2)(n+1)} = \frac{2a_n}{n+2}$$

Note that setting a_0 determines $a_2,\ a_4$, ... and setting a_1 determines $a_3,\ a_5$, ... We then have two undetermined coefficients a_0 and a_1 as we would expect from previous work. It's a second order equation, so we expect two arbitrary constants.

5.4.8 First solution

A simple choice to get one solution would be to set $a_0 = 1$ and $a_1 = 0$. Then all the odd index terms will be zero and

$$\begin{aligned} a_2 &= a_0 = 1 \\ a_4 &= \frac{2a_2}{4} = \frac{1}{2} \\ a_6 &= \frac{2a_4}{6} = \frac{1}{2\cdot 3} \end{aligned}$$

By induction we can show that

$$a_{2n} = \frac{1}{2 \cdot 3 \cdots n} = \frac{1}{n!}$$

Thus one solution is

$$y_1(t) = 1 + t^2 + \frac{t^4}{2} + \frac{t^6}{3!} + ... = e^{t^2}$$

We only know this is e^{t^2} because we recognize the series as being similar to that of e^x.

5.4.9 Second solution

Another simple choice to get one solution would be to set $a_0 = 0$ and $a_1 = 1$. Then all the even index terms will be zero and

$$\begin{aligned} a_3 &= \frac{2}{3}a_1 = \frac{2}{3} \\ a_5 &= \frac{2}{5}a_3 = \frac{2}{5}\frac{2}{3} \\ a_7 &= \frac{2}{7}a_5 = \frac{2}{7}\frac{2}{5}\frac{2}{3} \end{aligned}$$

By induction we can show that

$$a_{2n+1} = \frac{2^n}{3 \cdot 5 \cdots (2n+1)}$$

Thus one solution is

$$y_2(t) = t + \frac{2t^3}{3} + \frac{2^2 t^5}{3 \cdot 5} + ... = \sum_{n=0}^{\infty} \frac{2^n t^{2n+1}}{3 \cdot 5 \cdots (2n+1)}$$

Note that we don't recognize the series here.

5.5 Dealing with singular points

5.5.1 Problem

Solutions of

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

often become very large or oscillate wildly in the neighbourhood of a singular point. In such a case our previous method usually fails.

5.6 Handling singular points continued

5.6.1 Idea

The Euler-Cauchy equation gives us a clue about what to do when dealing with equations with singular points.

Example. Consider the equation

$$2ty'' + y' + ty = 0$$

This has a singular point at $t = 0$. Show that this is a regular singular point.

$$P(t) = 2t \qquad Q(t) = 1 \qquad R(t) = t$$

$$\begin{aligned} \frac{Q(t)}{P(t)} &= \frac{1}{2t} \qquad \therefore \text{ singular point at } t = 0 \\ t\frac{Q(t)}{P(t)} &= \frac{1}{2} \\ t^2\frac{R(t)}{P(t)} &= \frac{t^2}{2} \qquad \therefore\ t = 0 \text{ is a regular singular point} \end{aligned}$$

5.6.2 Series solution

We try a solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$$

which gives

$$\begin{aligned} y'(t) &= \sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1} \\ y''(t) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2} \end{aligned}$$

5.6.3 Substitution

Substituting back into the equation and factoring out the t^r we get

$$t^r\left[2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n-1} + \sum_{n=0}^{\infty}(n+r)a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^{n+1}\right] = 0$$

which we can rewrite as

$$t^r\left[2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n-1} + \sum_{n=0}^{\infty}(n+r)a_n t^{n-1} + \sum_{n=2}^{\infty} a_{n-2} t^{n-1}\right] = 0$$

and then writing the $n = 0$ and $n = 1$ terms explicitly

$$[2r(r-1)a_0 + ra_0]t^{r-1} + [2(1+r)ra_1 + (1+r)a_1]t^r + \sum_{n=2}^{\infty}[2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2}]t^{n+r-1} = 0$$

5.6.4 Recursion relations

Setting the coefficients of each power of t equal to zero gives

$$\begin{aligned} 2r(r-1)a_0 + ra_0 &= r(2r-1)a_0 = 0 \\ 2(r+1)ra_1 + (r+1)a_1 &= (r+1)(2r+1)a_1 = 0 \\ 2(n+r)(n+r-1)a_n + (n+r)a_n &= (n+r)[2(n+r)-1]a_n = -a_{n-2} \end{aligned}$$

5.6.5 Determining r

The first equation is called the indicial equation. It gives $r = 0$ or $r = 1/2$. We don't want $a_0 = 0$ ever because this corresponds to t^r in the sum. i.e. if it were, we'd just start the sum at the first nonzero coefficient.

5.6.6 Determining a_1

Once we have r, the second equation implies that $a_1 = 0$. This is true for either value of r in this case although in general if a value of r makes this equation zero, a_1 could be arbitrary.

$$\begin{aligned} a_6 &= -\frac{a_4}{6\cdot 11} = \frac{-a_0}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11} \\ a_{2n} &= (-1)^n \frac{a_0}{2\cdot 4\cdot 6\cdots(2n)\cdot 3\cdot 7\cdot 11\cdots(4n-1)} \\ &= \frac{(-1)^n a_0}{2^n\cdot n!3\cdot 7\cdot 11\cdots(4n-1)} \end{aligned}$$

5.6.7 Recursion relation

There are two possibilities for the recursion relation. For $r = 0$

$$a_n = -\frac{a_{n-2}}{n(2n-1)}$$

As a_1 is zero the odd coefficients are all zero. The even coefficients are

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 3} \\ a_4 &= -\frac{a_2}{4 \cdot 7} = \frac{a_0}{2 \cdot 4 \cdot 3 \cdot 7} \\ &\vdots \end{aligned}$$

5.6.8 One solution

We can get one solution by setting $a_0 = 1$. Then

$$y_1(t) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n n! 3 \cdot 7 \cdots (4n-1)}$$

is one solution.

5.6.9 Second recursion relation

If we take $r = 1/2$ then

$$\begin{aligned} a_n &= \frac{-a_{n-2}}{\left(n+\frac{1}{2}\right)\left[2\left(n+\frac{1}{2}\right)-1\right]} \\ a_n &= \frac{-a_{n-2}}{n(2n+1)} \end{aligned}$$

for n greater than or equal to 2. All of the odd coefficients are zero and the even coefficients are

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 5} \\ a_4 &= -\frac{a_2}{4 \cdot 9} = \frac{a_0}{2 \cdot 4 \cdot 5 \cdot 9} \\ a_6 &= -\frac{a_4}{6 \cdot 13} = \frac{-a_0}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 9 \cdot 13} \\ &\vdots \\ a_{2n} &= \frac{(-1)^n a_0}{2 \cdot 4 \cdot 6 \cdots (2n) \cdot 5 \cdot 9 \cdot 13 \cdots (4n+1)} \\ &= \frac{(-1)^n a_0}{2^n n! 5 \cdot 9 \cdot 13 \cdots (4n+1)} \end{aligned}$$

5.6.10 Second solution

By setting $a_0 = 1$ we get a second solution

$$y_2(t) = t^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{2^n n! 5 \cdot 9 \cdots (4n+1)}\right]$$

5.6.11 General solution

The general solution is given by taking a linear combination of the two solutions.

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

This is why we were allowed to choose a_0 in each case. We need any two linearly independent solutions.

5.7 Method of Frobenius

The method illustrated in the previous example is called the method of Frobenius. We discuss some complications.

5.8 Equal roots

5.8.1 Problem

If we have two possibilities for the exponent r then we can get two independent solutions to the ODE. What if we only have one solution?

5.8.2 Solution

If $y_1(t)$ is a solution then

$$y_2(t) = y_1(t)\ln(t) + t^r \sum_{n=1}^{\infty} b_n t^n$$

is also a solution. The b_n have to be solved for.

5.9 Roots differing by an integer

If the roots of the indicial equation differ by an integer then we could have a problem. Why?
If

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+r} \qquad \text{and} \qquad y_2(t) = \sum_{n=0}^{\infty} b_n t^{n+r+m} = \sum_{n=m}^{\infty} b_{n-m} t^{n+r}$$

then

$$c_1 y_1(t) + c_2 y_2(t) = \sum_{n=0}^{m-1} c_1 a_n t^{n+r} + \sum_{n=m}^{\infty} (c_1 a_n + c_2 b_{n-m}) t^{n+r}$$

This is only one solution, not two. i.e. the second series “catches” up with the first.

5.9.1 Solution

The solutions are of the form

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+r_1}$$

$$y_2(t) = A y_1(t)\ln(t) + \sum_{n=1}^{\infty} b_n t^{n+r_2}$$

5.10 Reduction of order

It may be easier to use reduction of order to get the second solutions in the case of repeated roots or roots differing by an integer. i.e. let $y_2(t) = v(t)y_1(t)$ to get

$$y_1 v'' + (q y_1' + p y_1) v' = 0$$

using reduction of order.

Example. Solve $ty'' + y = 0$.

$$\begin{aligned}
y &= a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \ldots \\
y' &= a_0 r t^{r-1} + a_1 (r+1) t^r + a_2 (r+2) t^{r+1} + \ldots \\
y'' &= a_0 r(r-1) t^{r-2} + a_1 (r+1) r t^{r-1} + a_2 (r+2)(r+1) t^r + \ldots \\
ty'' + y &= a_0 r(r-1) t^{r-1} + a_1 (r+1) r t^r + a_2 (r+2)(r+1) t^{r+1} + \ldots \\
&\quad + a_0 t^r + a_1 t^{r+1} + \ldots
\end{aligned}$$

We thus have

$$\begin{aligned}
r(r-1)a_0 &= 0 \qquad \to r = 0, 1 \\
(r+1)ra_1 + a_0 &= 0 \\
(r+2)(r+1)a_2 + a_1 &= 0 \\
(r+n+1)(r+n)a_{n+1} + a_n &= 0 \\
a_{n+1} &= \frac{-a_n}{(r+n+1)(r+n)} \\
a_1 &= \frac{-a_0}{(r+1)r} \\
a_2 &= \frac{-a_1}{(r+2)(r+1)} = \frac{a_0}{r(r+1)^2(r+2)} \\
a_3 &= \frac{-a_2}{(r+3)(r+2)} = \frac{-a_0}{r(r+1)^2(r+2)^2(r+3)} \\
a_n &= \frac{(-1)^n a_0}{r(r+1)^2(r_2)^2 \cdot \ldots \cdot (r+n)} \\
y_1 &= a_0 t + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 t^{n+1}}{1^2 2^2 \cdot \ldots \cdot (n-1)^2 n} = a_0 t + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 t^{n+1}}{n(n-1)!^2} \\
y_2 &= y_1 \ln|t| + t^0 \left(c_0 + \sum_{n=1}^{\infty} c_n t^n \right)
\end{aligned}$$

We then put this into the original differential equation and find the coefficients c_n. We won't be able to find a regular pattern, but we can get the first few terms (check these).

Thus the solution is

$$y_2 = y_1 \ln|t| + 1 - a_0 t - \frac{5a_0 t}{2^2} t^2 - \frac{12}{2^3 \cdot 3^3} a_0 t^3 - \ldots$$

Exercises: 1. $xy' = 3y + 3$ (Answer: $y = 1 + a_1 x$.)
2. $y' = 2xy$ (Answer: ($y = a_0 e^{x^2}$.)
3. Find the first six terms of $(x+1)y' = (2x+3)y$.
4. $(x-3)y' = xy$
5. $(1-x^4)y' = 4x^3 y$
6. $y'' - y = x$
7. Find the interval of convergence for
a. $\sum_{n=0}^{\infty} x^{2n}$
b. $\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}(x-5)^n$
c. $\sum_{n=0}^{\infty} n^{2n} x^n$
8. Rewrite $\sum_{n=2}^{\infty} \frac{n(n+1)}{n^2+1} x^{n-1}$ so the sum starts at 0 and find the interval of convergence.