Question 1. [4 points] Calculate

$$\textbf{a)} \int_{-2}^{2} \frac{1}{y^2-9}\,dy \qquad \textbf{b)} \int_{-\pi/2}^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\,dx$$

Solution. a) We have

$$\frac{1}{y^2-9} = \frac{1}{(y-3)(y+3)} = \frac{a}{y-3} + \frac{b}{y+3} = \frac{(a+b)y+3(a-b)}{(y-3)(y+3)},$$

with $a+b=0$ and $3(a-b)=1$, which implies $a=1/6$ and $b=-1/6$.

$$\begin{aligned}\int_{-2}^{2} \frac{1}{y^2-9}\,dy &= \frac{1}{6}\left(\int_{-2}^{2} \frac{1}{y-3}\,dy - \int_{-2}^{2} \frac{1}{y+3}\,dy\right)\\ &= \frac{1}{6}\left([\ln|y-3|]_{-2}^{2} - [\ln|y+3|]_{-2}^{2}\right)\\ &= -\frac{1}{3}\ln(5)\end{aligned}$$

b) Substituting $y=\cos(x)$, we have $dy = -\sin(x)dx$.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\,dx &= \int_{\cos(-\pi/2)}^{\cos(\pi/2)} -\frac{1}{1+y^2}\,dy\\ &= \int_{0}^{0} -\frac{1}{1+y^2}\,dy\\ &= [-\arctan(y)]_0^0\\ &= 0\end{aligned}$$

Question 2. [3 points] Solve the differential equation

$$\frac{dy}{dt} = \frac{6t\sin t}{y}$$

with initial condition $y(0)=5$.

Solution. First we separate the differential equation, putting the y's on the left and the t's on the right, and add an integral sign to obtain

$$\int y\,dy = \int 6t\sin t\,dt.$$

The left-hand integral is simple: $y^2/2$. (We omit the $+C$ with this integral.)

The right-hand integral requires integration by parts. Set $u=t$, $dv=\sin t\,dt$, so that $du=dt$, $v=-\cos t$. Then we obtain

$$\int 6t\sin t\,dt = 6\int t\sin t\,dt$$

$$= 6\left[-t\cos t - \int(-\cos t)\,dt\right]$$
$$= 6\left[-t\cos t + \sin t\right] + C.$$

So the general solution to the differential equation is

$$y^2/2 = 6\left[-t\cos t + \sin t\right] + C;$$

solving for y gives

$$y = \pm\sqrt{12(-t\cos t + \sin t) + C}. \tag{*}$$

(The constant C here is not the same as in the previous line. To be more accurate, you can write $+2C$ or $+D$ instead of $+C$ in equation (*), with D a different arbitrary constant.) This means that the various particular solutions are obtained by selecting one of $\pm$ and a particular constant C.

In our case we're given the initial condition $y(0) = 5$, that is, when t is 0, y takes the value 5. Plugging in $t = 0$ into equation (*) gives

$$y(0) = \pm\sqrt{12(0+0) + C} = \pm\sqrt{C}$$

So $\pm\sqrt{C} = 5$. Clearly, to achieve the value of 5, we must select the positive square root. Squaring both sides gives $C = 25$. So the solution to our differential equation with initial condition is

$$\boxed{y = \sqrt{12(-t\cos t + \sin t) + 25}}.$$

Question 3. [4 points] Evaluate the integral

$$\int \frac{x^3 + 2x^2 - 18x + 2}{x^2 + x - 12}\,dx.$$

Solution. $P(x) = x^3 + 2x^2 - 18x + 2$, and $Q(x) = x^2 + x - 12$. Then $\deg(P) = 3 > 2 = \deg(Q)$. So long division is necessary here. Using long division, we find that

$$x^3 + 2x^2 - 18x + 2 = (x+1)(x^2 + x - 12) + (-7x + 14)$$

and

$$\frac{x^3 + 2x^2 - 18x + 2}{x^2 + x - 12} = (x+1) + \frac{-7x + 14}{x^2 + x - 12}$$

Again, we notice that $x^2 + x - 12 = (x+4)(x-3)$. We need to find out two numbers A and B so that

$$\frac{-7x + 14}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3} = \frac{A(x-3) + B(x+4)}{(x+4)(x-3)} = \frac{(A+B)x + (-3A + 4B)}{(x+4)(x-3)}$$

Comparing the coefficients of the numerators, we obtain

$$A + B = -7,$$

$$-3A + 4B = 14.$$

Solving implies that $A = -6$ and $B = -1$. Finally,

$$\int \frac{x^3 + 2x^2 - 18x + 2}{x^2 + x - 12} dx = \int \left[(x+1) + \frac{-6}{x+4} + \frac{-1}{x-3}\right] dx = \frac{1}{2}x^2 + x - 6\ln|x+4| - \ln|x-3| + C.$$

Question 4. [6 points] For each of the following improper integrals, determine whether it converges, and determine its value if it does.

a) $\displaystyle\int_1^3 \frac{1}{t\,\ln t}\,dt$ **b**) $\displaystyle\int_0^\infty \frac{3}{18+2t^2}\,dt$ **c**) $\displaystyle\int_5^\infty \frac{\ln x}{x^2}\,dx$

Solution. a) Using substitution, we have

$$\begin{aligned} u &= \ln t \\ \frac{du}{dt} &= \frac{1}{t} \\ dt &= t du \end{aligned}$$

Then

$$\begin{aligned} \int_1^3 \frac{dt}{t\ln t} &= \lim_{\epsilon \to 1^+} \int_{t=\epsilon}^{t=3} \frac{1}{u} du \\ &= \lim_{\epsilon \to 1^+} [\ln|u|]_{t=\epsilon}^{t=3} \\ &= \lim_{\epsilon \to 1^+} [\ln|\ln t|]_\epsilon^3 \\ &= \lim_{\epsilon \to 1^+} \ln|\ln 3| - \ln|\ln \epsilon| \\ &= \infty, \end{aligned}$$

since when ϵ is close to (but larger than) 1, $\ln\epsilon$ is close to, but larger than, 0, so $|\ln\epsilon| = \ln\epsilon$ and $\ln|\ln\epsilon|$ approaches $-\infty$. Thus the integral diverges.

b) **Solution 1:**

$$\begin{aligned} \int_0^\infty \frac{3}{18+2t^2}\,dt &= \lim_{T\to\infty} \int_0^T \frac{3}{18+2t^2}\,dt \\ &= \lim_{T\to\infty} \frac{3}{18} \int_0^T \frac{1}{1+t^2/9}\,dt \\ &= \lim_{T\to\infty} \frac{1}{6} \int_0^T \frac{1}{1+(t/3)^2}\,dt \end{aligned}$$

Let $u = t/3$ so $dt = 3du$. Then

$$\int_0^\infty \frac{3}{18+2t^2}\,dt = \lim_{T\to\infty} \frac{1}{2} \int_{t=0}^{t=T} \frac{1}{1+u^2}\,du$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \frac{1}{2} \arctan u \bigg|_{t=0}^{t=T} \\
&= \lim_{T\to\infty} \frac{1}{2} \arctan(t/3) \bigg|_{0}^{T} \\
&= \lim_{T\to\infty} \frac{1}{2} \arctan(T/3) - \frac{1}{2} \arctan(0) \\
&= \frac{1}{2} \cdot \frac{\pi}{2} \\
&= \frac{\pi}{4}
\end{aligned}
$$

Solution 2: This time we'll make use of the formula

$$
\int \frac{dt}{t^2 + a^2} = \frac{1}{a} \arctan\left(\frac{t}{a}\right) + C, \qquad (*)
$$

a slightly easier variant of the formula presented in class. Let's start with the indefinite integral, and start by pulling out constant factors:

$$
\int \frac{3}{18 + 2t^2}\, dt = \frac{3}{2} \int \frac{dt}{9 + t^2} = \frac{3}{2} \int \frac{dt}{t^2 + 3^2} = \frac{1}{2} \arctan\left(\frac{t}{3}\right) + C,
$$

where we used (*) for the last equality. Putting in the limits for our improper integral gives:

$$
\begin{aligned}
\int_0^\infty \frac{3}{18 + 2t^2}\, dt = \lim_{T\to\infty} \int_0^T \frac{3}{18 + 2t^2}\, dt &= \lim_{T\to\infty} \frac{1}{2} \arctan\left(\frac{t}{3}\right) \bigg|_0^T \\
&= \frac{1}{2} \lim_{T\to\infty} (\arctan(T/3) - 0) = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.
\end{aligned}
$$

c)

$$
\int_5^\infty \frac{\ln x}{x^2}\, dx = \lim_{T\to\infty} \int_5^T \frac{\ln x}{x^2}\, dx.
$$

Using substitution, we have

$$
\begin{aligned}
u &= \ln x \\
\frac{du}{dx} &= \frac{1}{x} \\
dx &= x du
\end{aligned}
$$

Then

$$
\int_5^\infty \frac{\ln x}{x^2}\, dx = \lim_{T\to\infty} \int_{x=5}^{x=T} \frac{u}{x}\, du
$$

We need to resubstitute. Since $u = \ln x$, it follows that $x = e^u$. Thus

$$\int_5^\infty \frac{\ln x}{x^2}\, dx = \lim_{T\to\infty} \int_{x=5}^{x=T} ue^{-u}\, du.$$

Next we use integration by parts and let $w = u$ and $v' = e^{-u}$. Thus

$$\begin{aligned}
\int_5^\infty \frac{\ln x}{x^2}\, dx &= \lim_{T\to\infty} \left[-ue^{-u} + \int e^{-u}\, du\right] \\
&= \lim_{T\to\infty} \left[-ue^{-u} - e^{-u}\right]_{x=5}^{x=T} \\
&= \lim_{T\to\infty} \left[-\ln x e^{-\ln x} - e^{-\ln x}\right]_5^T \\
&= \lim_{T\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_5^T \\
&= \lim_{T\to\infty} \left[-\frac{\ln T}{T} - \frac{1}{T}\right] - \left[-\frac{\ln 5}{5} - \frac{1}{5}\right] \\
&= \lim_{T\to\infty} \left[-\frac{1/T}{1} - \frac{1}{T}\right] - \left[-\frac{\ln 5}{5} - \frac{1}{5}\right] \qquad \text{(using L'Hôpital's rule)} \\
&= [0 - 0] - \left[-\frac{\ln 5}{5} - \frac{1}{5}\right] \\
&= \frac{\ln 5}{5} + \frac{1}{5}
\end{aligned}$$

Therefore, the integral converges.

Question 5. [4 points] Define functions f and g by

$$f(x) = 3x^2 \qquad\qquad g = -3x + 6$$

a) Show that f and g intersect at points $x = -2$ and $x = 1$.
b) Calculate the area between f and g in the interval $-4 \le x \le 2$.

Solution. a) The points of intersection of f and g satisfy

$$f(x) - g(x) = 3x^2 + 3x - 6 = 3(x^2 + x - 2) = 3(x - 1)(x + 2) = 0.$$

Thus $x_1 = 1$ and $x_2 = -2$ are the intersection points.

b) We have $a < -2$ and $b > 1$. We need to calculate the area between f and g on the interval $[a, b]$. This is

$$\int_a^b |f(x) - g(x)| dx = \int_a^b |3x^2 + 3x - 6| dx = 3 \int_a^b |(x - 1)(x + 2)| dx.$$

The function $(x - 1)(x + 2)$ is negative for $-2 < x < 1$. Thus

$$\int_a^b |f(x) - g(x)| dx$$

$$
\begin{aligned}
&= 3\left(\int_a^{-2}(x-1)(x+2)dx + \int_{-2}^{1} -(x-1)(x+2)dx + \int_1^b (x-1)(x+2)dx\right)\\
&= 3\left(\int_a^{-2}(x^2+x-2)\,dx - \int_{-2}^{1}(x^2+x-2)\,dx + \int_1^b (x^2+x-2)\,dx\right)\\
&= 3\left(\left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)\Big|_a^{-2} - \left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)\Big|_{-2}^{1} + \left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)\Big|_1^{b}\right)\\
&= 3\left(2\left(-\frac{8}{3}+2+4\right) - \left(\frac{a^3}{3}+\frac{a^2}{2}-2a\right) - 2\left(\frac{1}{3}+\frac{1}{2}-2\right) + \left(\frac{b^3}{3}+\frac{b^2}{2}-2b\right)\right)\\
&= 27 + 3\left(-\frac{a^3}{3}-\frac{a^2}{2}+2a+\frac{b^3}{3}+\frac{b^2}{2}-2b\right).
\end{aligned}
$$

Substituting the values for a and b into the expression, we obtain

$$\int_{-4}^{2} |f(x)-g(x)|dx = \boxed{45}.$$

Question 6. [4 points] **a**) A very skinny 2m-long snake has density $\rho(x)$ kg/m at a distance x from the head of the snake, where

$$\rho(x) = 2x - x^2.$$

What is the total mass of the snake?
b) A less skinny 2m-long snake's body is given by rotating the function

$$y = \frac{e^{-x/2}}{10}, \qquad 0 \le x \le 2$$

about the x-axis (the units of y is also m). What is the volume of this snake?

Solution. a) The total mass of the snake is

$$\int_0^2 \rho(x)dx = \int_0^2 (2x-x^2)dx = \big(x^2 - \frac{x^3}{3}\big)\big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ kg}.$$

b) The volume of this snake is

$$
\begin{aligned}
&\int_0^2 \pi\big[y(x)\big]^2 dx = \int_0^2 \pi\left[\frac{e^{-\frac{x}{2}}}{10}\right]^2 dx\\
&= \int_0^2 \pi\frac{e^{-x}}{100}dx = \frac{\pi}{100}\int_0^2 e^{-x}dx = \frac{\pi}{100}(-e^{-x})\big|_0^2 = \frac{\pi}{100}(-e^{-2}+e^0) = \frac{\pi}{100}(1-e^{-2}) \text{ m}^3.
\end{aligned}
$$

Question 7. [5 points] Determine the average value of $f(x) = x\ln(x)$ over the range $0 \le x \le 2$. (Hint: to calculate an indeterminate limit, rearrange and use L'Hôpital's rule.)

Solution. The average value of $x\ln(x)$ over this interval is given by the quotient

$$\frac{\int_0^2 x\ln(x)\,dx}{2-0} = \frac{1}{2}\int_0^2 x\ln(x)\,dx. \tag{*}$$

This integral is improper because $\ln(x)$ is undefined at $x=0$; that means it is given as a limit of definite integrals:

$$\begin{aligned}
\int_0^2 x\ln(x)\,dx &= \lim_{a\to 0^+}\int_a^2 x\ln(x)\,dx \\
&= \lim_{a\to 0^+}\left(\frac{x^2}{2}\ln x\Big|_a^2 - \int_a^2 \frac{x^2/2}{x}\,dx\right) \\
&= \lim_{a\to 0^+}\left(\frac{x^2}{2}\ln x\Big|_a^2 - \int_a^2 \frac{x}{2}\,dx\right) \\
&= \lim_{a\to 0^+}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_a^2 \\
&= (2\ln 2 - 1) - \lim_{a\to 0^+}\left(\frac{a^2}{2}\ln a - \frac{a^2}{4}\right) \\
&= (2\ln 2 - 1) - \frac{1}{2}\lim_{a\to 0^+} a^2\ln a,
\end{aligned}$$

where we applied integration by parts with $u=\ln x$, $dv = x\,dx$, and in the last line, recognized that as a approaches 0, so will $a^2/4$.

Evaluating the remaining limit requires L'Hôpital's rule. We write the expression as a fraction in the most convenient way so that we can apply L'Hôpital:

$$a^2\ln a = \frac{\ln a}{a^{-2}}.$$

As a approaches 0 (from the positive side), both numerator and denominator become infinite — the numerator approaches $-\infty$ and the denominator approaches ∞ — so the condition of L'Hôpital's rule is met and we can conclude that

$$\lim_{a\to 0^+}\frac{\ln a}{a^{-2}} = \lim_{a\to 0^+}\frac{1/a}{-2a^{-3}} = \lim_{a\to 0^+} -\frac{a^2}{2} = 0,$$

where we first applied L'Hôpital and then simplified. So in the end, $\lim_{a\to 0^+} a^2\ln a = 0$, so the improper integral has value $2\ln 2 - 1$, and the average value of the function is given by equation (*) as $\boxed{(2\ln 2 - 1)/2}$