[4] 1. (a) Sketch the graph $y = |x-2| - |x|$.

Case i) $x \geq 2$

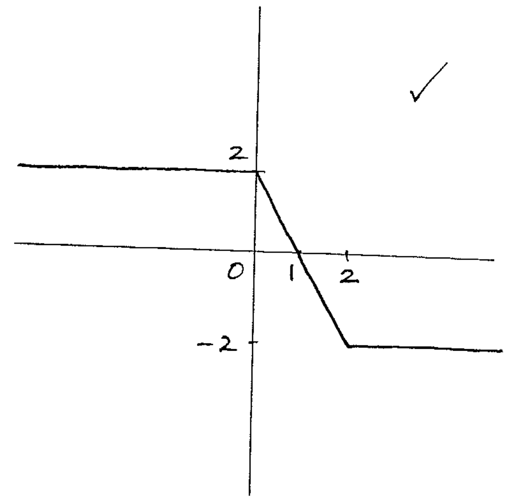
$y = x - 2 - x = -2$ ✓

Case ii) $0 \leq x < 2$

$y = -(x-2) - x = -2x + 2$ ✓

Case iii) $x < 0$

$y = -(x-2) - (-x) = 2$ ✓



✓

[4] (b) The Heaviside function H is the piecewise defined function given by the formula:

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

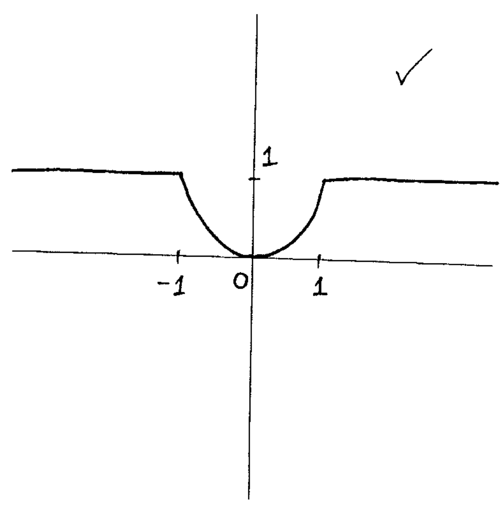
Sketch the graph $y = 1 + (x^2 - 1)H(1 - x^2)$.

Case i) $1 - x^2 \geq 0 \Leftrightarrow -1 \leq x \leq 1$

$y = 1 + (x^2 - 1) \cdot 1 = x^2$

Case ii) $1 - x^2 < 0 \Leftrightarrow x < -1$ or $x > 1$

$y = 1 + (x^2 - 1) \cdot 0 = 1$



✓

2. Let $\arcsin x = \sin^{-1} x$, $\arctan x = \tan^{-1} x$ denote the inverse sine and the inverse tangent functions, respectively.

[2] (a) State the domain and the range of $f(x) = -\arcsin\left(\frac{x}{2}\right)$.

$-1 \le \frac{x}{2} \le 1 \Rightarrow -2 \le x \le 2 \Rightarrow$ Domain of f is $[-2, 2]$ ✓

$y = -\arcsin\left(\frac{x}{2}\right) \Rightarrow -y = \arcsin\left(\frac{x}{2}\right) \Rightarrow -\frac{\pi}{2} \le -y \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow$ Range of f is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ✓

[3] (b) Determine the formula for the inverse function $f^{-1}(x)$, where $f(x)$ is the function in part (a).

$y = -\arcsin\left(\frac{x}{2}\right)$

$\sin y = \sin\left(-\arcsin\left(\frac{x}{2}\right)\right) \overset{✓}{=} -\sin\left(\arcsin\left(\frac{x}{2}\right)\right) \overset{✓}{=} -\frac{x}{2}$

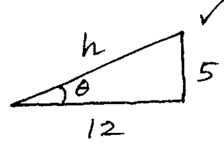
$x = -2\sin y$

✓ $f^{-1}(x) = -2\sin x$

[2] (c) Solve the equation $\tan(\arcsin x) = \frac{5}{12}$.

Let $\theta = \arcsin x \Leftrightarrow \sin\theta = x$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Since $\tan\theta = \frac{5}{12}$ ✓



$h = \sqrt{12^2 + 5^2} = 13$

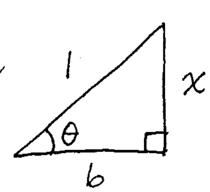
✓ $x = \sin\theta = \frac{5}{h} = \frac{5}{13}$

[2] (d) If $0 < x < 1$, show that $\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$.

Let $\theta = \arcsin x \Leftrightarrow \sin\theta = x$ and $0 < \theta < \frac{\pi}{2}$

Form a right triangle with $\sin\theta = \frac{x}{1} = x$.

✓



$b = \sqrt{1^2 - x^2} = \sqrt{1-x^2}$

✓ $\tan\theta = \frac{x}{\sqrt{1-x^2}}$ Hence $\arcsin x = \theta = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$

[5] 3. (a) Let $f(x) = 4 - \frac{x}{2}$. Using the precise ϵ, δ definition of limit, prove that $\lim_{x\to 2} f(x) = 3$.

Given $\varepsilon > 0$, we need to find $\delta > 0$ such that

✓ $$0 < |x-2| < \delta \Rightarrow |f(x) - 3| < \varepsilon .$$

Note that $|f(x)-3| < \varepsilon \iff |4 - \frac{x}{2} - 3| < \varepsilon$

$\iff |1 - \frac{x}{2}| < \varepsilon$ ✓

$\iff |\frac{x}{2} - 1| < \varepsilon$

$\iff |\frac{1}{2}(x-2)| < \varepsilon$ ✓

$\iff \frac{1}{2}|x-2| < \varepsilon$

$\iff |x-2| < 2\varepsilon$ ✓

✓ Thus we can let $\delta = 2\varepsilon$.

[2] (b) For every function $f(x)$ defined on $\mathbb{R}$, prove that the function $G(x) = f(x) - f(-x)$ is always odd.

$$G(-x) = f(-x) - f(-(-x)) \overset{\checkmark}{=} f(-x) - f(x)$$
$$\overset{\checkmark}{=} -(f(x) - f(-x)) = -G(x)$$

[5] 4. Prove the Limit Sum Law: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, then

$$\lim_{x\to a}[f(x)+g(x)] = L+M.$$

✓ Given $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

✓ $$\begin{cases} 0<|x-a|<\delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2} \\ 0<|x-a|<\delta_2 \Rightarrow |g(x)-M| < \frac{\varepsilon}{2} \end{cases}$$

✓ Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$0<|x-a|<\delta \Rightarrow 0<|x-a|<\delta_1 \text{ and } 0<|x-a|<\delta_2$$

$$\Rightarrow |f(x)-L| < \frac{\varepsilon}{2} \text{ and } |g(x)-M| < \frac{\varepsilon}{2}$$

✓ $$\Rightarrow |f(x)-L| + |g(x)-M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow |(f(x)+g(x)) - (L+M)| = |(f(x)-L)+(g(x)-M)|$$

✓ $$\leq |f(x)-L| + |g(x)-M| < \varepsilon$$

$$\Rightarrow |f(x)+g(x) - (L+M)| < \varepsilon$$

5. Evaluate the following limits if they exist. Explain your reasoning for each part.

[3] (a) $\lim_{x \to -\infty} \frac{(2-3x)^2}{2x^2 - x + 1}$ ✓ $= \lim_{x \to -\infty} \frac{9x^2 - 12x + 4}{2x^2 - x + 1}$

✓ $= \lim_{x \to -\infty} \frac{9 - \frac{12}{x} + \frac{4}{x^2}}{2 - \frac{1}{x} + \frac{1}{x^2}} = \frac{9 - 0 + 0}{2 - 0 + 0}$ ✓ $= \frac{9}{2}$

[3] (b) $\lim_{x \to 0} \left(\frac{1}{x\sqrt{1+x}} - \frac{1}{x} \right)$ ✓ $= \lim_{x \to 0} \frac{1 - \sqrt{1+x}}{x\sqrt{1+x}}$

$= \lim_{x \to 0} \frac{(1-\sqrt{1+x})(1+\sqrt{1+x})}{x\sqrt{1+x}\,(1+\sqrt{1+x})} = \lim_{x \to 0} \frac{1-(1+x)}{x\sqrt{1+x}\,(1+\sqrt{1+x})}$

✓ $= \lim_{x \to 0} \frac{-1}{\sqrt{1+x}(1+\sqrt{1+x})} = \frac{-1}{\sqrt{1}\,(1+\sqrt{1})}$ ✓ $= -\frac{1}{2}$

[4] (c) $\lim_{x \to 3} \frac{|5-2x| - |x-2|}{x^2 - 2x - 3}$

As $x \to 3$, $5 - 2x \to -1$ so $5 - 2x < 0$

$x - 2 \to 1$ so $x - 2 > 0$

$\|$ ✓✓

$\lim_{x \to 3} \frac{-(5-2x) - (x-2)}{x^2 - 2x - 3}$ ✓ $= \lim_{x \to 3} \frac{x-3}{(x-3)(x+1)} = \lim_{x \to 3} \frac{1}{x+1}$ ✓ $= \frac{1}{4}$

[4] 6. (a) Using the definition of the derivative, compute $f'(0)$ for the function $f(x) = x|x|$.

$$f'(0) \overset{\checkmark}{=} \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \overset{\checkmark}{=} \lim_{x\to 0} \frac{x|x|-0}{x} \overset{\checkmark}{=} \lim_{x\to 0} |x| \overset{\checkmark}{=} 0$$

Alternatively,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-0}{h} = \lim_{h\to 0} \frac{h|h|}{h}$$

$$= \lim_{h\to 0} |h| = 0$$

[3] (b) Without any explanation, state what $\lim_{x\to 0} \frac{\sin x}{x}$ is. With explanation, compute $\lim_{x\to 0} \frac{\sin^2(3x)}{x^2}$ using the previous limit.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\frac{\sin^2(3x)}{x^2} = \left[\frac{\sin(3x)}{x} \right]^2 = \left[\frac{\sin(3x)}{3x} \cdot 3 \right]^2$$

As $x \to 0$, $3x \to 0$ as well.

$$\lim_{x\to 0} \frac{\sin^2(3x)}{x^2} = \left[3 \cdot \lim_{x\to 0} \frac{\sin(3x)}{3x} \right]^2 = [3\cdot 1]^2 = 9 \quad \checkmark$$

[3] 7. (a) Find all horizontal tangent lines to the graph $y = \frac{2x^2}{x-8}$.

$$y' = \frac{4x(x-8) - 2x^2}{(x-8)^2} = \frac{2x^2 - 32x}{(x-8)^2} \quad ✓$$

$$y' = 0 \iff 2x^2 - 32x = 2x(x-16) = 0$$

$$\iff x = 0 \text{ or } x = 16 \quad ✓$$

When $x=0$, $y=0$

When $x=16$, $y = \frac{2 \cdot 16^2}{16-8} = \frac{512}{8} = 64$

Horizontal tangent lines are $y=0$ and $y=64$. ✓

[4] (b) Find the equation of the tangent line to the curve given by the equation $y = \frac{1}{\sin x + \cos x}$ at $(0, 1)$.

$$y' = \frac{-(\cos x - \sin x)}{(\sin x + \cos x)^2} \quad ✓✓$$

At $x=0$, $y' = \frac{-(1-0)}{(0+1)^2} = -1$ ✓

Tangent line at $(0,1)$ is given by

$$y - 1 = (-1)(x-0)$$

$$y = -x + 1 \quad ✓$$

8. (a) Let

[4]

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x^2}\right) & \text{if } x > 0, \\ e^x + c & \text{if } x \le 0. \end{cases}$$

Find the value of the constant c that makes $f(x)$ continuous everywhere.

$f(0) = e^0 + c = 1 + c = \lim_{x \to 0^-} f(x)$ ✓

✓✓ Since $-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$, $-x \le x\sin\left(\frac{1}{x^2}\right) \le x$ when $x > 0$.
Since $\lim_{x\to 0^+}(\pm x) = 0$, $\lim_{x \to 0^+} f(x) = \lim_{x\to 0^+} x\sin\left(\frac{1}{x^2}\right) = 0$
by Squeeze Theorem.

In order for $f(x)$ to be continuous at $x=0$, we need

$1 + c = 0$

$c = -1$ ✓

[3] (b) Show that the following equation has at least one real solution:

$$\log_2(x+1) = \sqrt[3]{1 - x^2}.$$

Write the name of any theorem that you are using and explain why you can use that theorem for the above equation.

Let $f(x) = \log_2(x+1) - \sqrt[3]{1-x^2}$

✓ $f(0) = \log_2 1 - \sqrt[3]{1} = 0 - 1 = -1 < 0$

✓ $f(1) = \log_2 2 - \sqrt[3]{0} = 1 - 0 = 1 > 0$

✓ $f(x)$ is continuous when $x > -1$, so $f(x)$ is continuous on the interval $[0, 1]$. Using Intermediate Value Theorem, we conclude that $f(x) = 0$ has at least one real solution.