Solutions to Test 2 - MATH 1107C - Winter 2010

Version 2

PART I: Multiple choice questions (3 points each)

Choose and circle only one answer. No partial marks here.
No justification is required.

1. The matrix that induces the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that represents a reflection through the y-axis (or x_2-axis) is

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution: **(c)**

2. If the homogeneous system $\begin{cases} x + y = 0 \\ ax + by = 0 \end{cases}$ has non-trivial solutions then

(a) $a = b$ **(b)** $a \neq b$ **(c)** $a = -b$ **(d)** $a \neq -b$

Solution: **(a)**

3. Let A, B and C be $n \times n$ matrices. Which of the following statements is true in general?
(i) $AB = BA$
(ii) $(A - B)(A + B) = A^2 - B^2$
(iii) $(A^T)^4 = (A^4)^T$
(iv) $(AB + C)^T = A^T B^T + C^T$.

(a) (i) only **(b)** (ii) only **(c)** (iii) only **(d)** (iv) only

Solution: **(c)**

4. Let A be a 4×4 matrix, B be a 4×2 matrix, and C be a 4×2 matrix. Then the size (or order) of $(3A - BC^T)^T$ is

(a) 2×2 **(b)** 4×2 **(c)** 2×4 **(d)** 4×4

Solution: **(d)**

5. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 0 & -1 \end{bmatrix}$. If $C = AB$ then the $(2,1)$-entry in C (c_{21}, second row and first column in C) is

(a) 8 **(b)** -2 **(c)** -1 **(d)** 1

Solution: **(a)**

PART II: Long answer questions
Show all your work.

6. [**13 points**] Express every solution to the following system as the sum of a specific solution plus a solution to the associated homogeneous system.

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 3 \\ x_1 + 2x_2 + 3x_4 = 7 \\ x_1 + 2x_2 + x_3 + 5x_4 = 11 \end{cases}$$

Solution: $\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 2 & 0 & 3 & 7 \\ 1 & 2 & 1 & 5 & 11 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 4 & 8 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2}$

$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 7 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 - 2r - 3s \\ r \\ 4 - 2s \\ s \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where $\begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \end{bmatrix}$ is a particular solution and $r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ is the solution to the associated homogeneous system.

7. [**10 points**] Find A such that

$$2\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - 3A^T \right) = \begin{bmatrix} 4 & 0 \\ 8 & 16 \end{bmatrix}.$$

Solution:

$$2\left(\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix} - 3A^T \right) = \begin{bmatrix} 4 & 0 \\ 8 & 16 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 6 \\ 2 & -2 \end{bmatrix} - 6A^T = \begin{bmatrix} 4 & 0 \\ 8 & 16 \end{bmatrix}$$

$$-6A^T = \begin{bmatrix} 4 & 0 \\ 8 & 16 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 2 & -2 \end{bmatrix}$$

$$-6A^T = \begin{bmatrix} 6 & -6 \\ 6 & 18 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$\text{Hence } A = (A^T)^T = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}.$$

8. Let $R_\pi \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the counterclockwise rotation about the origin through π radians (that is, 180^o).

(a) [**5 points**] Show that R_π is induced by a matrix and find the matrix.

(b) [**2 points**] Is the matrix found in (a) symmetric? Why or why not?

Solution: (a) $R_\pi(x, y) = (-x, -y)$, and so $R_\pi(x, y) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$.

Thus, the induced matrix is $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ where the first column is $R_\pi(1, 0)$ and the second column is $R_\pi(0, 1)$.

(b) The matrix A is symmetric, since

$$A^T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = A.$$