April 19, 2012

Midterm MAT1332D
Winter 2012

Study guide - Final Exam

As you read these let me know if you think I have missed anything. I will add more as I can.

The best way to study is:

1. make sure you can do all of the problems of the homework assignments and midterms;
2. try a couple problems similar to each of these;
3. make up a problem of each type on your own.

The last step is important as it will give you a good feeling for the different ways in which a problem could appear.

Let's recall what we did this semester. The following topics and problems showed up on the first midterm:

- Integrals: indefinite, definite, pure-time differential equations. (Look in main text for examples.)

  To solve these problems you need to know the *fundamental theorem of calculus* and the techniques of *substitition*, *integration by parts*, and *partial fractions*.

  1. Find $\int P(x)e^x dx$, where $P(x) = 3x^2 + x$.

     Use integration by parts. Let

$$u = 3x^2 + x \qquad dv = e^x dx$$
$$du = 6x + 1dx \qquad v = e^x$$

$$\int (3x^2 + x)e^x dx = (3x^2 + x)e^x - \int (6x + 1)e^x dx$$

     Use integration parts again. Let

$$r = 6x + 1 \qquad ds = e^x dx$$
$$dr = 6dx \qquad s = e^x$$

$$\begin{aligned} \int (3x^2 + x)e^x dx &= (3x^2 + x)e^x - \int (6x + 1)e^x dx \\ &= (3x^2 + x)e^x - (6x + 1)e^x + \int 6e^x dx \\ &= (3x^2 - 5x + 5)e^x + c \end{aligned}$$

2. Find $\int_{-1}^{1} = (3x^2 + x)e^x dx$, where $P(x) = 3x^2 + x$ and $a = -1$, $b = 1$.

$$\begin{aligned}
\int_a^b &= P(x)e^x dx \\
&= (3x^2 - 5x + 5)e^x \Big|_{-1}^{1} \\
&= (3 - 5 + 5)e - (3(-1)^2 - 5(-1) + 5)e^{-1} \\
&= 3e - 13e^{-1}
\end{aligned}$$

3. Solve the pure-time differential equation

$$\frac{dx}{dt} = \sin(3t)$$

with initial condition $x(0) = 1$.
Integrate

$$\int dx = \int \sin(3t)dt$$

$$x(t) = -\frac{1}{3}\cos(3t) + c$$

$$1 = x(0) = -\frac{1}{3}\cos(3 \cdot 0) + c$$

$$\frac{4}{3} = c$$

$$x(t) = -\frac{1}{3}\cos(3t) + \frac{4}{3}$$

Note: There are not too many different types of integration by parts problems. You should see if you can write an example of each type we have seen. Also, try the above problem with other choices of polynomials $P(x)$ and limits of integration a and b.

- Improper integrals (Look in main text, midterms and homeworks for examples.)

  To solve these you need the techniques from above, the definitions of improper integrals, and some knowledge of limits. The problems we have generally seen ask you to determine whether a given integral converges or diverges.

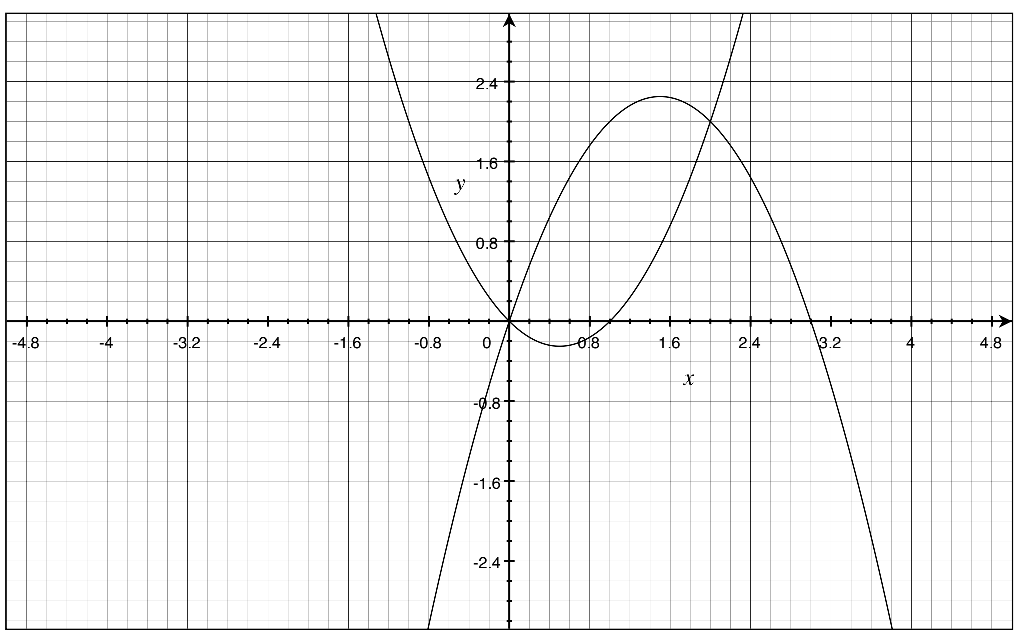
- Riemann sums (Look in main text for examples.)

  Given a function on an interval and a number of subintervals to divide the interval into, then you should be able to calculate left- and right-hand Riemann sums.

- Area between curves

  These problems often follow three steps:
  1) Find points of intersection of the graphs of two curves.
  2) Calculate which function has a greater value on each interval.
  3) Integrate the appropriate difference of the functions.

  Many of the questions we have seen consider two polynomial functions over an interval $[0, a]$. So to generate practice problems we can choose functions with no constant terms, for example:
  $$f(x) = x^2 - x \text{ and } g(x) = 3x - x^2$$
  The reason I chose polynomials with no constant terms is that it implies
  $$f(0) = 0 = g(0)$$
  so we know the curves intersect at the origin. Next, we determine the second point of intersection by setting the functions equal
  $$x^2 - x = 3x - x^2$$
  which can be rewritten as a quadratic polynomial equation
  $$2x^2 - 4x = 0$$
  or
  $$2x(x - 2) = 0.$$
  So we know that $x = 2$ is also a point of intersection and we will integrate the difference of f and g over the interval $[0, 2]$. The graphs look like



We can see that $g(x) > f(x)$, but we can also check algebraically by choosing a point in the interval, say $x = 1$, and calculating

$$(1) = 1^2 - 1 = 0 < 2 = 3(1) - (1)^2 = g(1),$$

So we have

$$\begin{aligned}
\int_0^2 (g(x) - f(x))dx &= \int_0^2 (3x - x^2) - (x^2 - x)dx \\
&= \int_0^2 -2x^2 + 4xdx \\
&= -\frac{2}{3}x^3 + 2x^2 \Big|_0^2 \\
&= -\frac{2}{3}(2)^3 + 2(2)^2 \\
&= \frac{8}{3}
\end{aligned}$$

Try choosing other functions for f and g to generate more practice problems. See what happens if you choose one of them to be a cubic polynomial or a trig function. Also, see how the problem changes if you choose polynomials with constant terms.

- Volumes of revolution (Course notes, p.3)

  Here you need to repeat steps 1) and 2) above and then remember the volume integration formula that comes from the area of a circle πr^2 (since at each x-value a slice of the solid will be a disk.) Then recall that the function you are integrating is essentially the radius. (**NOTE**: These are the problems where it matters if you integrate $f^2 - g^2$ or $(f - g)^2$. Make sure you know which one is correct.)

  A good practice problem is to use the techniques of volumes of revolution to find the formula for the volume of a sphere with a given radius.

- Separable differential equations

  These problems often follow three steps:
  1) Separate the variables. (This may involve some factoring)
  2) Integrate each side of the resulting equation.
  3) Solve for the state variable.

  If there are initial conditions given then you would also solve for the constant of integration.

- Word problems (**Note**: These are often separable differential equation problems given in paragraph form.

Let's recall what we did this semester after the second midterm:

- Studying autonomous differential equations (Look in main text for examples.)

  We learned techniques to study differential equations that we can not necessarily describe solutions to explicitly. These techniques included: *equilibrium points*, *stability analysis*, and *phase-line diagrams*.

- Complex numbers (Course notes, p.14)

  First we learned the *modulus*, *inverse*, and *complex conjugate*. We learned to write complex numbers in the form $a + bi$. This is often done by rationalizing. We also learned to transfer from this form to polar coordinates and back. We also learned to plot complex numbers in the plane.

- Matrix operations (Course notes, p.29)

  Addition and multiplication with matrices, finding the determinant, the inverse and the transpose.

- Systems of equations and row reduction (Course notes, p.20)

  You need to know how change a system of equations into a matrix or augmented matrix and back. Further, you need be able to quickly row reduce a matrix. (**Note**: There is some expectation of speed when testing row reduction. You should practice row reducing random matrices until it becomes second nature.)

  The forms you need are *row-echelon form* and *reduced row-echelon form*.

- Eigenvalues and eigenvectors (Course notes, p.44)

  Find eigenvalues of 2×2- and 3×3-matrices. You may sometimes be asked to use the characteristic equation for 2×2-matrices.

  Find eigenvectors associated to real and complex eigenvalues. This is done using row-reduction. (**Note**: Given a matrix A and a possible eigenvalue λ, if you are asked to find an eigenvector for λ, and you find that $\det(A - \lambda I) \neq 0$, then λ can't be an eigenvalue. This is a good way to check your work.)

- Solving for variables in matrices and systems of equations.

  These are problems where you are asked to solve for a variable, say a, as an entry of a matrix so that the matrix is invertible.

  Another form we have have seen is to find values of variables such that a certain system of equations has a particular number of solutions.

Finally, the topics we covered after the second midterm.

- Functions of two variables (Course notes, p.56)

  Find the *domain* and *range* of a function, draw *level curves*. Find *partial derivatives*.

- *Linear approximations* and *tangent planes* to surfaces. (Course notes, p.57)

  Here you need the generalization of point-slope form to two dimensions and to use partial derivatives to determine the gradient.

  Consider the function
  $$f(x,y) = \cos(xy)$$
  from $\mathbb{R}^2$ to $\mathbb{R}$. The domain of cosine is all real numbers, so any pair (x,y) is in the domain of f, so $\mathrm{dom}(f) = \mathbb{R}^2$. The range of cosine is $[-1,1]$, so $\mathrm{ran}(f) = [-1,1]$.

  Now let's consider the point $\left(\pi, \frac{1}{3}\right)$ and find the tangent plane to f at this point.

  We compute the partial derivatives
  $$\frac{\partial f}{\partial x} = -y\sin(xy) \qquad \frac{\partial f}{\partial y} = -x\sin(xy)$$
  and the values
  $$f\left(\pi, \frac{1}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$
  $$\frac{\partial f}{\partial x}\left(\pi, \frac{1}{3}\right) = -\frac{\sin\left(\frac{\pi}{3}\right)}{3} = -\frac{\sqrt{3}}{6}$$
  $$\frac{\partial f}{\partial y}\left(\pi, \frac{1}{3}\right) = -\pi\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\pi$$
  The equation for the tangent plane is
  $$z - \frac{1}{2} = -\frac{\sqrt{3}}{6}(x-\pi) - \frac{\sqrt{3}}{2}\pi\left(y - \frac{1}{3}\right).$$

- Solve *systems of differential equations* with initial conditions. (Course notes, p.70)

  Need to know the *general form of solutions of systems of linear differential equations.*

  Consider the system of linear differential equations
  $$\frac{dx}{dt} = x(t) + 2y(t)$$
  $$\frac{dy}{dt} = 3x(t) + 2y(t)$$
  (There are several good problems in the textbook which will also give you practice with word problems. One example is *Newton's law of cooling.*)

The solutions have the general form:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2.$$

The first step is to find the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

The characteristic equation is

$$\lambda^2 - 3\lambda - 4 = 0$$

and the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$.
Next, we find the eigenvectors. For $\lambda_1 = 4$ we have

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$$

so we apply $R_1 + R_2 \to R_2$ to obtain

$$\begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}$$

We set the free variable $y = t$ and have

$$-3x + 2t = 0$$

$$x(t) = \frac{2}{3}t$$

Setting $t = 1$, we have the eigenvector

$$v_1 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Similarly, we find

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{4t} \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Often the problems also include *initial conditions*. Let's use $x(0) = 1$ and $y(0) = 0$. Then we can solve for C_1 and C_2 as well. Continuing the problem above we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = C_1 e^0 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 e^0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}C_1 - C_2 \\ C_1 + C_2 \end{bmatrix}$$

which is just the system of linear equations

$$\begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This can be solved by row-reduction or by finding the inverse matrix. I will use the inverse

$$\begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

We have

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix}$$

and

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{3}{5} e^{4t} \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} - \frac{3}{5} e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

- Phase portrait - sketching solution curves

  We can use the techniques of sketching solutions for systems of linear or non-linear equations. Often these problems will involve linear differential equations since we spent more time analyzing the phase portrait in this case. You need to find *nullclines*, *equilibrium points*, *stability*, *direction arrows*, and *approximate sketches of solution curves*. In the non-linear case, the stability analysis requires the Jacobian matrix since there are multiple equlibrium points.

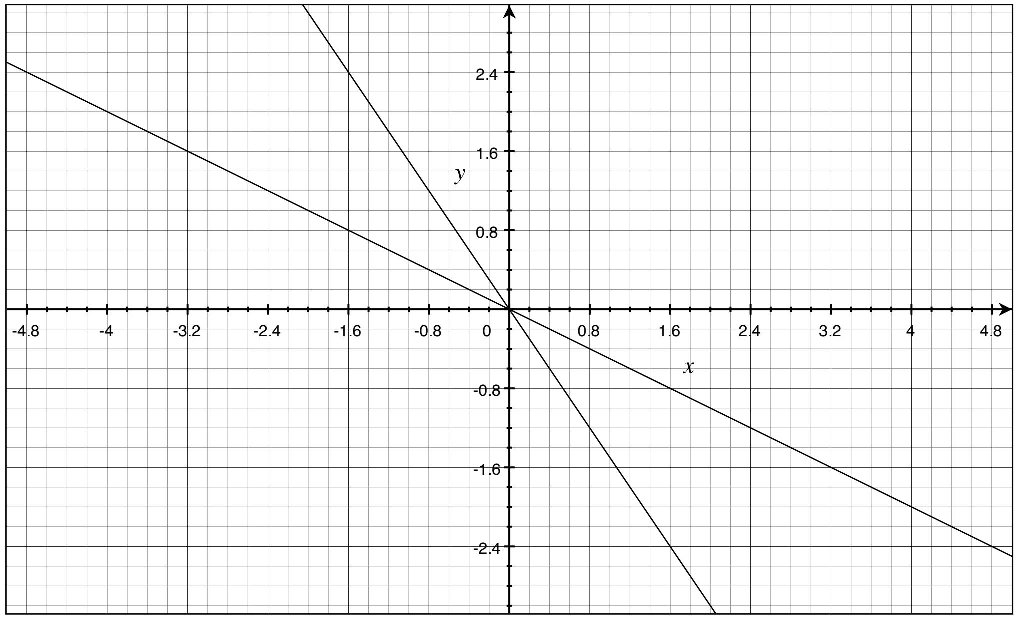
  Suppose now that we had not been able to find the explicit solution to the system of differential equations above. Then we might want to find information about the solutions so that we can sketch an approximate solution.

  First we find the nullclines. The x-nullcline is

$$y = -\frac{1}{2}x$$

  and the y-nullcline is

$$y = -\frac{3}{2}x.$$



The only equilibrium point is $(0,0)$. We already found the eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -1$ so we know $(0,0)$ is *unstable*.

We can draw *direction arrows* on the nullclines. We plug the point $\left(\frac{3}{2}, -\frac{3}{4}\right)$ on the x-nullcline into the expression

$$\frac{dy}{dt} = 3\frac{3}{2} + 2(-\frac{3}{4}) = 3 > 0,$$

so the direction of flow is positive in the y-direction and the arrow points towards the origin.

Other points on the nullclines can be given direction arrows similarly. You should draw one arrow on each of the four rays formed by the intersection of the nullclines.

We can also draw direction arrows at any point in the regions between the nullclines. For example, plugging the point $(1,0)$ given in the initial conditions into the differential equations we get the vector

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

which tells us that from the point $(1,0)$ the system moves along the line with slope $\frac{3}{1} = 3$ and passing through $(1,0)$. This is moving away from the origin as we should expect since the origin is an unstable equilibrium point.

To continue sketching a curve we can choose a small increment to move along this line (vector) from the point $(1,0)$. At the new point we can repeat this process of finding a direction arrow and continue this process to obtain a piece-wise approximation of the solution curve.

In case of *systems of non-linear differential equations* you should be able to determine the *Jacobian matrix* and use it to find stability of equilibrium points.