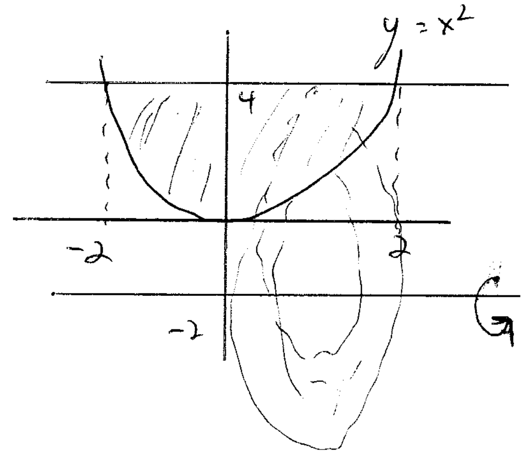
1. [4 points] The finite region bounded by curves $y = x^2$ and $y = 4$ is rotated about the line $y = -2$. Sketch the region. Find the volume of the resulting solid.

Work.



Washer Method

$$V = \pi \int_{-2}^{2} \left((4 - -2)^2 - (x^2 - -2)^2 \right) dx$$

$$= \pi \int_{-2}^{2} \left(6^2 - (x^2+2)^2 \right) dx$$

$$= \pi \int_{-2}^{2} \left(-x^4 - 4x^2 + 32 \right) dx$$

$$= \pi \left(\frac{-x^5}{5} - \frac{4}{3}x^3 + 32x \right) \Big|_{-2}^{2} \cong 402.256$$

2. [4 points] An object is taken from an oven at a temperature of 300 ° C to a room at 20 ° C. Its temperature $u = u(t)$ then decreases according to Newton's law of cooling $\frac{du}{dt} = k(u-20)$. After 15 minutes its temperature is 200 ° C.

(a)Find the temperature $u(t) =$ $\boxed{20 + 280\, e^{\frac{t \ln(9/14)}{15}}}$

(b)When will the temperature reach 100 ° C? $t =$ $\boxed{42.53 \text{ m}}$

(c)Sketch the graph of $u(t)$ showing the values for $t = 0$ and as $t \to \infty$.

Work.

a)

$$\frac{du}{dt} = k(u-20) \Rightarrow \int \frac{du}{u-20} = \int k\, dt$$

$$\Rightarrow + \ln|u-20| = kt + C \Rightarrow |u-20| = e^{kt} e^{C}$$

$$\Rightarrow u = 20 + D e^{kt} \quad \text{where } D = \pm e^{C} \text{ or } 0.$$

$$u(0) = 300 = 20 + D \Rightarrow D = 280$$

$$\Rightarrow u(t) = 20 + 280\, e^{kt}$$

$$u(15) = 200 = 20 + 280\, e^{15k} \Rightarrow \frac{180}{280} = e^{15k}$$

$$\Rightarrow \frac{9}{14} = e^{15k} \Rightarrow 15k = \ln(9/14)$$

$$\Rightarrow k = \frac{1}{15}\ln(9/14)$$

$$u(t) = 20 + 280\, e^{t \ln(9/14)/15}$$

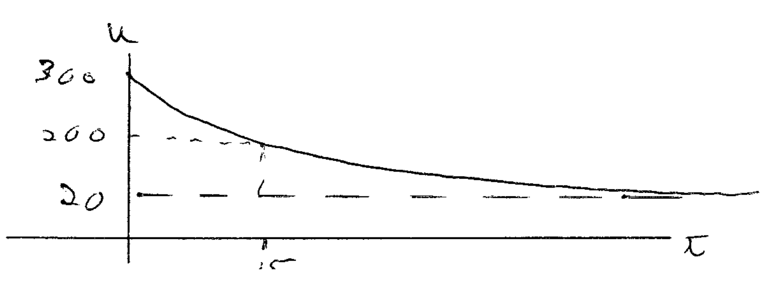
b)

$$100 = u(t) = 20 + 280\, e^{t \ln(9/14)/15}$$

$$\Rightarrow \frac{80}{280} = e^{t \ln(9/14)/15} \Rightarrow \frac{2}{7} = e^{t \ln(9/14)/15}$$

$$\Rightarrow \ln\left(\frac{2}{7}\right) = \frac{t}{15} \ln\left(\frac{9}{14}\right) \Rightarrow t = \frac{15 \ln(2/7)}{\ln(9/14)} \doteq 42.53 \text{ m}.$$

c)



← not to scale

3. [4 points] Determine if the following series are convergent or divergent and state the name of the test you used for that purpose. (Record you answer next to the series below.)

	conv./div.	test used
(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$	div	int. test
(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n^3+5)^{1/3}}$	conv.	comp. test

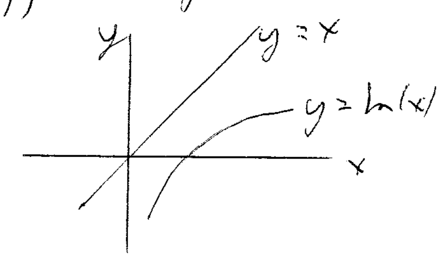
Work.

a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ consider $f(x) = \dfrac{1}{x\ln(x)}$ for $x \geq 2$

① $f(x) > 0$ for $x \geq 2$

② $f(x)$ is decreasing for $x \geq 2$

because $f'(x) = \dfrac{\ln(x) - x}{(x\ln(x))^2} < 0$ for $x \geq 2$



③ $\displaystyle\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$

$$= \lim_{N\to\infty} \int_2^N \frac{1}{x\ln(x)}\,dx = \lim_{N\to\infty} \int_{\ln(2)}^{\ln(N)} \frac{1}{u}\,du$$

$u = \ln(x)$

$du = \frac{1}{x}dx$

$$= \lim_{N\to\infty} \ln(u)\Big|_{\ln(2)}^{\ln(N)} = \lim_{N\to\infty} \left(\ln(\ln(N)) - \ln(\ln(2))\right) = \infty$$

$\Rightarrow$ Series diverges by integral test

b) $n(n^3+5)^{1/3} > n(n^3)^{1/3} = n^2 \Rightarrow \dfrac{1}{n(n^3+5)^{1/3}} < \dfrac{1}{n^2}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, (integral test with $p = 2 > 1$),

then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n^3+5)^{1/3}}$ converges by the comparison test

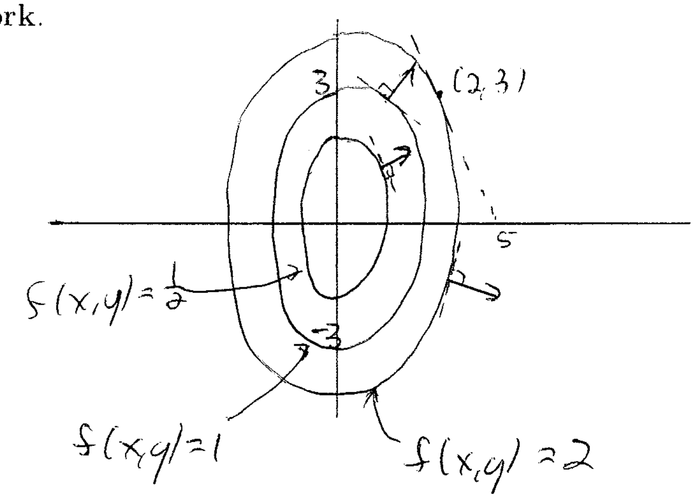
4. [4 points] Let $f(x,y) = \dfrac{x^2}{4} + \dfrac{y^2}{9}$.

(a)Sketch at least three labeled level curves of f and the gradient vector at a point on each of them. (All in one sketch, on the same axes.)

(b)Find the equation of the tangent plane to the graph of f at the point where $(x,y) = (2,3)$.

Work.

a)



b)

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{x}{2}, \frac{2y}{9}\right)$$

$$\nabla f(2,3) = \left(1, \frac{2}{3}\right)$$

equation of the tangent plane at $(2,3,2)$

$$z = f(2,3) + \frac{\partial f}{\partial x}(2,3)\,(x-2) + \frac{\partial f}{\partial y}(2,3)\,(y-3) = 0$$

$$z = 2 + (x-2) + \frac{2}{3}(y-3) = 0$$

$$z = -2 + x + 2y$$

5. [2 points] Evaluate $\int_0^3 \frac{2}{(x-2)^{4/3}}\,dx$ if possible.

A. -10.76 B. 1.24 C. -1.24 D. 2.40 E. -2.40 F. divergent

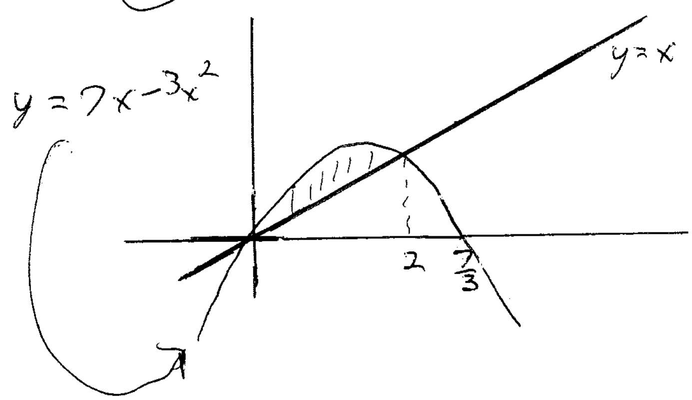
$$\int_0^3 \frac{2}{(x-2)^{4/3}}\,dx = \int_0^2 \frac{2}{(x-2)^{4/3}}\,dx + \int_2^3 \frac{2}{(x-2)^{4/3}}\,dx$$

$$= \lim_{\alpha \to 2^-} \int_0^\alpha \frac{2}{(x-2)^{4/3}}\,dx + \lim_{\alpha \to 2^+} \int_\alpha^3 \frac{2}{(x-2)^{4/3}}\,dx$$

$$= \underbrace{\lim_{\alpha \to 2^-} -6\,(x-2)^{-1/3}\Big|_0^\alpha}_{\to +\infty} + \underbrace{\lim_{\alpha \to 2^+} -6\,(x-2)^{-1/3}\Big|_\alpha^3}_{\to +\infty}$$

6. [2 points] Find the area of the region enclosed by the curves $y = x$ and $y = 7x - 3x^2$.

A. 2 B. 4 C. 6 D. 8 E. 9 F. 11



$y = 7x - 3x^2 = (7 - 3x)\,x$

intersection

$x = y = 7x - 3x^2$

$\Rightarrow 0 = 6x - 3x^2$

$= 3x(2 - x)$

$\Rightarrow x = 0$ and 2

$$A = \int_0^2 (7x - 3x^2 - x)\,dx$$

$$= \int_0^2 (6x - 3x^2)\,dx = (3x^2 - x^3)\Big|_0^2 = 4$$

7. [2 points] Consider the initial value problem $y' = x + y$, $y(0) = 1$. Use Euler's method with step size 0.1 to approximate $y(0.2)$.

A. 1.2000 B. 1.1000 C. 1.2200 D. 1.2300 E. 1.0100 F. 1.0200

$y_{n+1} = y_n + f(x_n, y_n)\,h$ where $f(x,y) = x + y$

$h = 0.1$

$x_0 = 0$

$y_0 = 1$

$y_1 = y_0 + 0.1\,(x_0 + y_0) = 1.1$

$y_2 = y_1 + 0.1\,(x_1 + y_1)$

$= 1.1 + 0.1\,(0.1 + 1.1)$

$= 1.22$

	x_n	y_n
0	0	1
1	0.1	1.1
2	0.2	

8. [2 points] Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{y}{1+x^2}$, $y(0) = 2$ to find $y(1)$.

A. 2.19 B. 2.00 C. 4.39 D. 4.00 E. 3.14 F. undefined

$\dfrac{dy}{dx} = \dfrac{y}{1+x^2} \Rightarrow \displaystyle\int \frac{dy}{y} = \int \frac{dx}{1+x^2}$

$\Rightarrow \ln|y| = \arctan(x) + C$

$\Rightarrow |y| = e^{\arctan(x)} e^{C}$

$\Rightarrow y = D e^{\arctan(x)}$ où $D = \pm e^{C}$ or 0

$y(0) = 2 \Rightarrow 2 = D e^{\arctan(0)} = D$

$\Rightarrow y = 2 e^{\arctan(x)}$

$y(1) = 2 e^{\arctan(1)} \approx 4.39$

9. [2 points] Let $f(x) = e^{x^3}$. Calculate the 9th derivative $f^{(9)}(0)$ using the exponential series and Taylor's formula.

A. 0 B. 350 C. 810 D. 3490 E. 60480 F. 78520

$$f(x) = e^{x^3} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}(0)\, x^n$$

Taylor's Series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\Rightarrow e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = f(x)$$

x^9 x^9

$$\frac{1}{9!}\frac{d^9 f}{dx^9}(0) = \frac{1}{3!} \Rightarrow \frac{d^9 f}{dx^9}(0) = \frac{9!}{3!} = 4\cdot5\cdot6\cdot7\cdot8\cdot9$$

$$= 60480$$

10. [2 points] If $(1+x^3)^{0.6} = \sum_{n=0}^{\infty} b_n x^n$ then $b_3 =$

A. −1.00 B. −0.05 C. 0.00 D. 0.05 E. 0.60 F. 1.00

$$(1+z)^{0.6} = \sum_{n=0}^{\infty} \binom{0.6}{n} z^n$$

Binomial Theorem

$$\Rightarrow (1+x^3)^{0.6} = \sum_{n=0}^{\infty} \binom{0.6}{n} x^{3n}$$

$$\Rightarrow b_3 = \binom{0.6}{1} = 0.6$$

The coefficient of x^3

11. [2 points] Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(-5)^n (x+3)^n}{n}$.

A. 1 B. 5 C. 1/5 D. 3 E. 1/3 F. ∞

$$\lim_{n\to\infty} \left(\frac{|-5|^{n+1} |x+3|^{n+1}}{n+1} \right) \Big/ \left(\frac{|-5|^n |x+3|^n}{n} \right)$$

$$= \lim_{n\to\infty} 5|x+3| \left(\frac{n}{n+1}\right) = \lim_{n\to\infty} 5|x+3| \left(\frac{1}{1+\underbrace{1/n}_{\to 0}}\right)$$

$$= 5|x+3| < 1 \Rightarrow |x+3| < \frac{1}{5}$$

radius of conv. $= \frac{1}{5}$

12. [2 points] Evaluate $\sum_{n=0}^{\infty} \frac{2^n + 3^{n+1}}{7^n}$.

A. divergent B. 3.15 C. 1.40 D. 7.35 E. 6.65 F. 5.25

$$\sum_{n=0}^{\infty} \frac{2^n + 3^{n+1}}{7^n} = \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n + 3 \sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n$$

$$= \frac{1}{1-\frac{2}{7}} + \frac{3}{1-\frac{3}{7}} = \frac{7}{5} + \frac{21}{4}$$

geometric series

$$= \frac{133}{20} = 6.65$$

13. [2 points] Determine which of the series converge.

$$(1) \sum_{n=1}^{\infty} \frac{(-1)^n 100}{n} \qquad (2) \sum_{n=1}^{\infty} \frac{n}{5^n} \qquad (3) \sum_{n=1}^{\infty} \frac{n^2}{(n+100)^2}$$

A. (1) and (2)   B. (1) and (3)   C. (2) and (3)

D. (1) only   E. (2)only   F. (3) only

(1) $100 \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ alt. series that converges because $\frac{1}{n} \to 0$

(2) $\sum_{n=1}^{\infty} \frac{n}{5^n}$ converges because $\lim_{n\to\infty} \frac{\frac{n+1}{5^{n+1}}}{\frac{n}{5^n}} = \lim_{n\to\infty} \frac{1}{5}\left(\frac{n+1}{n}\right) = \frac{1}{5} < 1$

(3) $\sum_{n=1}^{\infty} \frac{n^2}{(n+100)^2}$ diverges because $\lim_{n\to\infty} \frac{n^2}{(n+100)^2} = 1 \neq 0$

14. [2 points] *Use the linear approximation* of $f(x,y) = 2x^2y^2 + 3xy + x$ at $(1,1)$ to approximate $f(0.9, 1.1)$.

A. 5.8302   B. 5.9000   C. 6.1000   D. 6.0302   E. 6.0000   F. 5.9302

$\frac{\partial f}{\partial x} = 4xy^2 + 3y + 1 \Rightarrow \frac{\partial f}{\partial x}(1,1) = 8$

$\frac{\partial f}{\partial y} = 4x^2y + 3x \Rightarrow \frac{\partial f}{\partial y}(1,1) = 7$

$\Rightarrow$

$f(x,y) \simeq f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1)$

$\Rightarrow f(0.9, 1.1) \simeq 6 + 8(0.9-1) + 7(1.1-1)$

$= 5.9$

15. [2 points] Find the equation of the tangent plane of the surface $z^2 + x^2 - 4xy + y^2 = 2$ at the point $(1, 1, 2)$.

A. $2z - x - y = 2$ B. $2z + x + y = 6$ C. $z + x + y = 4$
D. $z = 2$ E. $2x + 2y - z = 2$ F. $z - x - y = 0$

Surface:
$$F(x,y,z) = x^2 - 4xy + y^2 + z^2 - 2 = 0$$

$$\nabla F(x,y,z) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (2x - 4y, 2y - 4x, 2z)$$

$$\Rightarrow \nabla F(1,1,2) = (-2, -2, 4)$$

Eqn. of tangent plane

$$\nabla F(1,1,2) \cdot (x-1, y-1, z-2) = -2(x-1) - 2(y-1) + 4(z-2) = 0$$

$$\Rightarrow -2x - 2y + 4z - 4 = 0$$

$$\Rightarrow -x - y + 2z - 2 = 0$$

16. [2 points] Let $f(x, y, z) = 2x^2y^2z + 4xy^3z^2$. Find $f_{xyz}(2, 1, 1)$.

A. 28 B. 32 C. 36 D. 24 E. 38 F. 40

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(2x^2y^2 + 8xy^3z\right)\right)$$

$$= \frac{\partial}{\partial x}\left(4x^2y + 24xy^2z\right)$$

$$= 8xy + 24y^2z$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z}(2,1,1) = 16 + 24 = 40$$

17. [2 points] Find the directional derivative of $f(x,y) = x^2y + 4y^2$ at the point $(2,1)$ in the direction of the vector $\langle 1, \sqrt{3} \rangle$.

A. $2 - 6\sqrt{3}$ B. $2 + 3\sqrt{3}$ C. $2 - 3\sqrt{3}$

D. $2 + 6\sqrt{3}$ E. $4 + 12\sqrt{3}$ F. $2 - \sqrt{3}$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2xy, x^2 + 8y)$$

$$\vec{u} = \frac{1}{\|(1,\sqrt{3})\|}(1,\sqrt{3}) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\frac{\partial f}{\partial \vec{u}}(2,1) = \nabla f(2,1) \cdot \vec{u} = (4, 12) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 2 + 6\sqrt{3}$$

18. [2 points] Suppose $z = f(x,y)$ where $x = g(t)$ and $y = h(t)$. Given the data

$g(1) = 1,\ g'(1) = 2,$
$h(1) = 2, h'(1) = 3,$
$f_x(1,2) = -1,\ f_y(1,2) = 2.$

Find $\frac{dz}{dt}$ when $t = 1$.

A. -1 B. 1 C. 4 D. -4 E. 8 F. impossible

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial f}{\partial x}(g(t), h(t))\, g'(t) + \frac{\partial f}{\partial y}(g(t), h(t))\, h'(t)$$

$$\Rightarrow \frac{dz}{dt}(1) = f_x(1,2) \cdot 2 + f_y(1,2) \cdot 3 = -2 + 6 = 4$$