MAT 1302 E: Mathematical methods II
Professor: Aziz Khanchi

Test III
November 2014

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–2) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.

(b) The number of points available for each question is indicated in square brackets.

(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(d) Write your student number at the top of each page in the space provided.

(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(f) You have to show your work for each question.

Good luck!

Question	1	2	3	4	5	6	Total
Maximum	1	1	8	9	4	7	30
Note							

Part 1: Multiple choice questions, no partial marks, choose the best possible answer.

Question 1. [**1 point**] Let $H = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \right\}$, $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x^2 + y^2 + z^2 \leq 1 \right\}$,

$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, $K = \left\{ \begin{bmatrix} 2b - 3c \\ b \\ c \end{bmatrix} \middle| b, c \in \mathbb{R} \right\}$, $G = \left\{ \begin{bmatrix} b^2 + c^2 \\ b \\ c \end{bmatrix} \middle| b, c \in \mathbb{R} \right\}$

List all the sets which are subspaces of $\mathbb{R}^3$?

A) H

B) H and G,

C) H and U,

D) H, K and G,

E) H and K.

Solution: (e)

Question 2. [**1 point**] Let

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & k+1 & 0 \\ -1 & -3 & h \end{bmatrix}.$$

.

For what values of h and k rank(A)= 3?

A) $k = -1$, $h \neq 2$,

B) $k = -3$, $h = 2$,

C) $k \neq -1$, $h \neq 2$,

D) $k \neq -3$, $h \neq -1$,

E) $k \neq 2$, $h = -1$.

Solution: (c)

Question 3. [**8 points**] We consider an economy with two sectors: Agriculture and Service. For each unit of output, Agriculture requires $\frac{1}{3}$ units from Agriculture and $\frac{1}{2}$ units from Service. For each unit of output, Service uses $\frac{1}{9}$ units from Agriculture and $\frac{5}{6}$ units from Service.

(a) [**1 point**] What is the consumption matrix C for this economy?

Solution: The consumption matrix C is

$$C = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{2} & \frac{5}{6} \end{bmatrix}$$

(b) [**2 points**] Compute $I - C$ and $(I-C)^{-1}$.

Solution:

$$I - C = \begin{bmatrix} 1-\frac{1}{3} & -\frac{1}{9} \\ -\frac{1}{2} & 1-\frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{9} \\ -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

Also, $(I-C)=\frac{2}{18} - \frac{1}{18} = \frac{1}{18}$. Therefore,

$$(I-C)^{-1} = \frac{1}{\frac{1}{18}} \begin{bmatrix} \frac{1}{6} & \frac{1}{9} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} = 18 \begin{bmatrix} \frac{1}{6} & \frac{1}{9} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 9 & 12 \end{bmatrix}.$$

(c) [**2 point**] Determine what intermediate demands are created if Agriculture plans to produce 30 units and Service plans to produce 18 units.

Solution: The intermediate demands will be

$$C \begin{bmatrix} 30 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{2} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 30 \\ 18 \end{bmatrix} = \begin{bmatrix} 12 \\ 30 \end{bmatrix}$$

(d) [**3 point**] Find the production levels, using the inverse of $(I-C)$, that will satisfy the final demand of 1 unit from Agriculture and 10 units from Service.

Solution: The final demand vector is $\vec{d} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$. Therefore, the production level necessary meet the final demand can be calculated as

$$\vec{x} = (I-C)^{-1} d = \begin{bmatrix} 3 & 2 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 10 \end{bmatrix} = \begin{bmatrix} 23 \\ 129 \end{bmatrix}$$

Question 4. [**9 points**] Let $A = \begin{bmatrix} 1 & 2 & -1 & 3 & -2 \\ 2 & 3 & 0 & 1 & -1 \\ 0 & 1 & -2 & 5 & -1 \\ 0 & 2 & -4 & 10 & -2 \end{bmatrix}$. The following matrix is the row-reduced echelon form of A:

$$\begin{bmatrix} 1 & 0 & 3 & -7 & 0 \\ 0 & 1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A) [**3 points**] Find a basis for $Col\ A$ (column space of A).

Solution: Pivot columns of A form a basis for the column space of A:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -1 \\ -2 \end{bmatrix} \right\}$$

B) [**3 points**] Find a basis for $Nul\ A$ (null space of A).

Solution: Here is the homogeneous linear system corresponding to the matrix:

$$\begin{aligned} x_1 + 3x_3 - 7x_4 &= 0 \\ x_2 - 2x_3 + 5x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

Therefore, the general solution is:

$$\begin{aligned} x_1 &= -3x_3 + 7x_4 \\ x_2 &= 2x_3 - 5x_4 \\ x_5 &= 0 \\ x_3, x_4 &: \text{free} \end{aligned}$$

The general solution in parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 + 7x_4 \\ 2x_3 - 5x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, a basis for the null space of A can be given as

$$\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

C) [**3 points**] Find the dimensions of $Col\ A$ and $Nul\ A$.

Solution: dim Col A=3, dim Nul A=2.

Question 5. [**4 points**] Let $z = 1 - i$ and $w = 3 + 5i$.

A) [**2 points**] Compute $\frac{w}{z}$. Your results should be in the form $a + bi$ where a and b are real numbers.

Solution:

$$\frac{w}{z} = \frac{w\overline{z}}{z\overline{z}} = \frac{(3+5i)(1+i)}{(1-i)(1+i)} = \frac{3+3i+5i+5(i)^2}{1+i-i-(i)^2} = \frac{3+3i+5i+5(-1)}{1+i-i-(-1)}$$
$$= \frac{(3-5)+(3+5)i}{1+1} = \frac{-2+8i}{2} = \frac{-2}{2} + \frac{8i}{2} = -1 + \frac{8}{2}i = -1 + 4i.$$

B) [**2 points**] compute $wz - |w + z - 1|$. Your results should be in the form $a + bi$ where a and b are real numbers.

Solution:

$$wz - |w+z-1| = (3+5i)(1-i) - |(3+5i)+(1-i)-1| = 8+2i-|4+4i-1|$$
$$= 8+2i-|(4-1)+4i| = 8+2i-|3+4i| = 8+2i-\sqrt{3^2+4^2}$$
$$8+2i-\sqrt{25} = 8+2i-5 = (8-5)+2i = 3+2i$$

Question 6. [7 points]

A) [5 points] Let

$$A = \begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & -3 & 1 & -2 \\ 3 & -2 & 17 & 0 \\ -1 & 5 & -1 & -2 \end{bmatrix}.$$

Compute det A by row reduction.

Solution:

$$\det A = \begin{vmatrix} 2 & -4 & 6 & -2 \\ 1 & -3 & 1 & -2 \\ 3 & -2 & 17 & 0 \\ -1 & 5 & -1 & -2 \end{vmatrix} \xlongequal{L_1 \leftrightarrow \frac{1}{2}L_1} 2 \begin{vmatrix} 1 & -2 & 3 & -1 \\ 1 & -3 & 1 & -2 \\ 3 & -2 & 17 & 0 \\ -1 & 5 & -1 & -2 \end{vmatrix}$$

$$\xlongequal{\substack{-L_1+L_2 \\ -3L_1+L_3 \\ L_1+L_4}} 2 \begin{vmatrix} 1 & -2 & 3 & -1 \\ 0 & -1 & -2 & -1 \\ 0 & 4 & 8 & 3 \\ 0 & 3 & 2 & -3 \end{vmatrix} \xlongequal{L_2 \leftrightarrow -L_2} -2 \begin{vmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 8 & 3 \\ 0 & 3 & 2 & -3 \end{vmatrix}$$

$$\xlongequal{\substack{-4L_2+L_3 \\ -3L_2+L_4}} -2 \begin{vmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -4 & -6 \end{vmatrix} \xlongequal{L_3 \leftrightarrow L_4} 2 \begin{vmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 2(1)(1)(-4)(-1) = 8$$

B) [1 points] Let $D = \begin{bmatrix} 1 & -3 & 0 & 0 \\ -3 & 7 & 5 & 0 \\ 9 & -1 & 3 & -8 \\ 10 & 0 & -4 & -2 \end{bmatrix}$. Compute the cofactor C_{32} for D. **Solution:**

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 & 0 \\ -3 & 5 & 0 \\ 10 & -4 & -2 \end{vmatrix} = (-1)(1)(5)(-2) = 10$$

C) [1 points] Suppose A, B and C are 2×2 matrices with det $A = 6$, det $B = -4$ and

$$AB^{-1}AC^T = 3AB.$$

Find det C.

Solution: The determinants of left and right sides are equal:

$$\det\,(AB^{-1}AC^{T}) = \det\,(3AB).$$

Using properties of determinants:

$$(\det\,A)(\det\,B^{-1})(\det\,A)(\det\,C^{T})=(\det\,3A)(\det\,B)=3^{2}(\det\,A)(\det\,B)$$

Alors,

$$(6)(\tfrac{-1}{4})(6)(\det\,C^{T}) = 3^{2}(6)(-4)$$

$\Longrightarrow$

$$\det\,C^{T} = \tfrac{(9)(6)(-4)}{(6)(6)(\frac{-1}{4})} = 24.$$

Finally, $\det\,C = \det\,C^{T} \Longrightarrow \det\,C = 24$.