

## Solutions to Test 2 - MATH 1107C - Winter 2010

Version 1

**PART I: Multiple choice questions (3 points each)**

Choose and circle only one answer. No partial marks here.  
No justification is required.

1. The matrix that induces the transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that represents a reflection through the  $x$ -axis (or  $x_1$ -axis) is

(a)  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$     (b)  $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$     (c)  $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$     (d)  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

**Solution:** (a)

2. If the homogeneous system  $\begin{cases} x - y = 0 \\ ax + by = 0 \end{cases}$  has non-trivial solutions then

(a)  $a = b$     (b)  $a \neq b$     (c)  $a = -b$     (d)  $a \neq -b$

**Solution:** (c)

3. Let  $A$ ,  $B$  and  $C$  be  $n \times n$  matrices. Which of the following statements is true in general?

- (i)  $AB = BA$
- (ii)  $(A - B)(A + B) = A^2 - B^2$
- (iii)  $(A^T)^4 = (A^4)^T$
- (iv)  $(AB + C)^T = A^T B^T + C^T$ .

(a) (i) only    (b) (ii) only    (c) (iii) only    (d) (iv) only

**Solution:** (c)

4. Let  $A$  be a  $5 \times 5$  matrix,  $B$  be a  $5 \times 3$  matrix, and  $C$  be a  $5 \times 3$  matrix. Then the size (or order) of  $(6A - BC^T)^T$  is

(a)  $3 \times 5$     (b)  $5 \times 3$     (c)  $3 \times 3$     (d)  $5 \times 5$

**Solution:** (d)

5. Let  $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 5 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 5 & 0 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$ . If  $C = AB$  then the  $(3, 2)$ -entry in  $C$  ( $c_{32}$ , third row and second column) is

(a) 0    (b) 23    (c) 21    (d) -1

**Solution:** (b)

**PART II: Long answer questions**
**Show all your work.**

**6. [13 points]** Express every solution to the following system as the sum of a specific solution plus a solution to the associated homogeneous system.

$$\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = -3 \\ x_1 + 3x_2 + 3x_4 = 2 \\ x_1 + 3x_2 + x_3 + 4x_4 = 7 \end{cases}$$

**Solution:**

$$\begin{aligned} & \left[ \begin{array}{cccc|c} 1 & 3 & -1 & 2 & -3 \\ 1 & 3 & 0 & 3 & 2 \\ 1 & 3 & 1 & 4 & 7 \end{array} \right] \xrightarrow{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1}} \left[ \begin{array}{cccc|c} 1 & 3 & -1 & 2 & -3 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 2 & 10 \end{array} \right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \\ & \left[ \begin{array}{cccc|c} 1 & 3 & -1 & 2 & -3 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[ \begin{array}{cccc|c} 1 & 3 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - 3r - 3s \\ r \\ 5 - s \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where  $\begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix}$  is a particular solution and  $r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$  is the solution to the associated homogeneous system.

**7. [10 points]** Find  $A$  such that

$$2 \left( \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} + 3A^T \right) = \begin{bmatrix} 2 & 0 \\ 4 & -6 \end{bmatrix}.$$

**Solution:**

$$\begin{aligned} 2 \left( \begin{bmatrix} 4 & 6 \\ -1 & 3 \end{bmatrix} + 3A^T \right) &= \begin{bmatrix} 2 & 0 \\ 4 & -6 \end{bmatrix} \\ \begin{bmatrix} 8 & 12 \\ -2 & 6 \end{bmatrix} + 6A^T &= \begin{bmatrix} 2 & 0 \\ 4 & -6 \end{bmatrix} \\ 6A^T &= \begin{bmatrix} 2 & 0 \\ 4 & -6 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -2 & 6 \end{bmatrix} \\ 6A^T &= \begin{bmatrix} -6 & -12 \\ 6 & -12 \end{bmatrix} \\ A^T &= \begin{bmatrix} -1 & -2 \\ 1 & -2 \end{bmatrix} \end{aligned}$$

$$\text{Hence } A = (A^T)^T = \begin{bmatrix} -1 & -2 \\ 1 & -2 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 \\ -2 & -2 \end{bmatrix}.$$

8. Let  $R_\pi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote the counterclockwise rotation about the origin through  $\pi$  radians (that is,  $180^\circ$ ).

(a) [5 points] Show that  $R_\pi$  is induced by a matrix and find the matrix.

(b) [2 points] Is the matrix found in (a) skew-symmetric? Why or why not?

**Solution:** (a)  $R_\pi(x, y) = (-x, -y)$ , and so  $R_\pi(x, y) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$ .

Thus, the induced matrix is  $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$  where the first column is  $R_\pi(1, 0)$  and the second column is  $R_\pi(0, 1)$ .

(b) The matrix  $A$  is not skew-symmetric, as the entries on the main diagonal are not zero. Another explanation is:

$$A^T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \neq -A.$$