

1. Consider the differential equation

$$Ly = 2x^2 y'' - xy' + (1+x^2)y = 0 \quad (1)$$

(a) Classify the points $0 \leq x < \infty$ as ordinary points, regular singular points, or irregular singular points.(b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

a) $x=0$ IS A SINGULAR POINT AND ALL $x: 0 < x < \infty$ ARE ORDINARY POINTSFOR $x=0$: $\lim_{x \rightarrow 0} x \left(\frac{-x}{2x^2} \right) = -\frac{1}{2} = p_0$ AND $\lim_{x \rightarrow 0} x^2 \left(\frac{1+x^2}{2x^2} \right) = \frac{1}{2} = q_0$. SINCE $|p_0|, |q_0| < \infty$ $x=0$ IS A REGULAR SINGULAR POINT.

b) CONSIDER THE CAUCHY EULER EQ:

$$x^2 y'' - \frac{x}{2} y' + \frac{y}{2} = 0$$

$$\text{LET } y = x^r \Rightarrow 2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (2r-1)(r-1) = 0 \Rightarrow r = \frac{1}{2}, 1.$$

$$c) y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$0 = Ly = 2x^2 y''$$

$$= \sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2}$$

$$= \sum_{m=0}^{\infty} a_m [(m+r)(2(m+r-1)-1)+1] x^{m+r} + \sum_{m=2}^{\infty} a_{m-2} x^{m+r}$$

$$0 = a_0 [2r(r-1)-r+1] x^r + a_1 [2(r+1)r-(r+1)+1] x^{r+1} + \sum_{m=2}^{\infty} [a_m [(m+r)(2(m+r-1)-1)+1] + a_{m-2}] x^{m+r}$$

$$x^r] a_0 [2r^2-3r+1] = a_0 (2r-1)(r-1) = 0 \quad r = \frac{1}{2}, r = 1 \text{ AS ABOVE}$$

$$x^{r+1}] a_1 [2r^2+r] = 0 \quad \text{IF } r = \frac{1}{2} \quad 2r^2+r = 1 \quad r = 1 \quad 2r^2+r = 3 \quad \text{IN BOTH CASES WE MUST HAVE } a_1 = 0$$

$$x^{m+r} \quad m \geq 2] \text{ RECURSION } a_m = -a_{m-2} / [(m+r)(2(m+r-1)-1)+1] \quad m \geq 2. \quad (*)$$

$$r = \frac{1}{2}: a_m = -a_{m-2} / [(m+\frac{1}{2})(2m+\frac{1}{2}-1)+1] = -a_{m-2} / [(2m+1)(m-1)+1] = \frac{-a_{m-2}}{m(2m-1)}$$

$$a_2 = -a_0 / 2.3 \quad a_4 = -a_2 / 4.7 = +a_0 / 168$$

$$\therefore y_0(x) = a_0 x^{1/2} [1 - x^2/16 + x^4/168 - \dots]$$

$$r = 1: a_m = -a_{m-2} / [(m+1)(2m-1)+1] = -a_{m-2} / m(2m+1)$$

$$a_2 = -a_0 / 2.5 \quad a_4 = -a_2 / 4.9 = +a_0 / 360$$

$$\therefore y_1(x) = a_0 x [1 - x^2/10 + x^4/360 - \dots]$$

NOTE SINCE $a_1 = 0$ THE RECURSION (*) IMPLIES $0 = a_3 = a_5 = \dots$

2. Consider a conducting metal bar of length $\pi/2$ whose initial temperature is $u(x, 0) = x$ and which loses heat to its surroundings. Assume that the left end of the bar is maintained at a zero temperature while the right end is insulated. The temperature distribution in the bar $u(x, t)$ is determined by the following initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= u_{xx} - u, & 0 < x < \pi/2, & \quad t > 0 \\ u(0, t) &= 0, & u_x(\pi/2, t) &= 0 \\ u(x, 0) &= x \end{aligned} \quad (2)$$

- (a) Determine the solution to the boundary value problem (2) by separation of variables.

[14 marks]

- (b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition $u_x(\pi/2, t) = 0$ with $O(\Delta x^2)$ accuracy.

Hint: Taylor's expansion may prove useful: $f(x + \Delta x) = f(x) + \frac{f'(x)}{1!} \Delta x + \frac{f''(x)}{2!} \Delta x^2 + O(\Delta x^3)$.

[6 marks]

[total 20 marks]

a) Let $u(x, t) = X(x)T(t)$ AND SUBSTITUTE INTO THE PDE:

$$X\dot{T} = X''T - XT \quad \text{WHERE } \dot{T}(t) = \frac{dT}{dt} \text{ \& } X'(x) = \frac{dX}{dx}$$

$\div XT$ $\frac{\dot{T}}{T} + 1 = \frac{X''}{X} = -\lambda^2$ A CONSTANT.

T $\dot{T} = -(1+\lambda^2)T \Rightarrow T(t) = Ce^{-(1+\lambda^2)t}$

X $X'' + \lambda^2 X = 0 \quad X(0) = 0 = X'(\pi/2)$

$$X = A \cos \lambda x + B \sin \lambda x \quad X' = -A\lambda \sin \lambda x + B\lambda \cos \lambda x$$

$$0 = X(0) = A \cdot 1 \Rightarrow A = 0; \quad 0 = X'(\pi/2) = B\lambda \cos(\lambda\pi/2) \Rightarrow \lambda_n = (2n+1) \quad n=0, 1, 2, \dots$$

$$X_n = \sin(2n+1)x$$

$$\therefore u(x, t) = \sum_{n=0}^{\infty} b_n e^{-(1+\lambda_n^2)t} \sin \lambda_n x$$

$$x = u(x, 0) = \sum_{n=0}^{\infty} b_n \sin(2n+1)x$$

$$\therefore b_n = \frac{2}{\pi/2} \int_0^{\pi/2} x \sin(2n+1)x \, dx = \frac{4}{\pi} \left[-x \frac{\cos(2n+1)x}{(2n+1)} \Big|_0^{\pi/2} + \frac{1}{(2n+1)} \int_0^{\pi/2} \cos(2n+1)x \, dx \right]$$

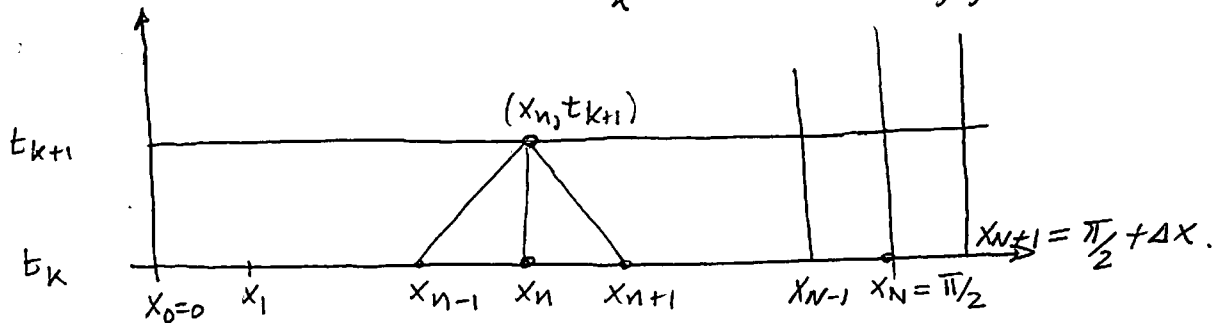
$$= \frac{4}{\pi} \left\{ \frac{\sin(2n+1)x}{(2n+1)^2} \Big|_0^{\pi/2} \right\} = \frac{4}{\pi} \frac{(-1)^n}{(2n+1)^2}$$

$$\therefore u(x, t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-[1+(2n+1)^2]t} \sin(2n+1)x$$

(Question 2 Continued)

DIVIDE THE SPATIAL DOMAIN $[0, \pi/2]$ INTO $N+1$ MESH POINTS

$$x_n = n \Delta x \quad n=0, 1, \dots, N \quad \Delta x = \pi/2N.$$

AND DEFINE THE TIME STEPS $t_k = \Delta t \cdot k \quad k=0, 1, \dots$ 

USING TAYLOR'S THM

$$u(x \pm \Delta x, t) = u(x, t) \pm \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{24} \frac{\partial^4 u}{\partial x^4} \pm \dots \quad (1 \pm)$$

$$\frac{(1+) + (1-)}{\Delta x^2} \Rightarrow \frac{u(x+\Delta x, t) - 2u(x, t) + u(x-\Delta x, t)}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2} + O(\Delta x^2)$$

$$u(x, t+\Delta t) = u(x, t) + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \dots$$

$$\therefore \frac{u(x, t+\Delta t) - u(x, t)}{\Delta t} = \frac{\partial u}{\partial t} + O(\Delta t)$$

$$\therefore u_t \approx \frac{u_n^{k+1} - u_n^k}{\Delta t} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2} - u_n^k$$

$$\therefore u_n^{k+1} = u_n^k + \frac{\Delta t}{\Delta x^2} (u_{n+1}^k - 2u_n^k + u_{n-1}^k) - \Delta t u_n^k$$

TO APPROXIMATE THE DERIVATIVE AT $x_N = \pi/2$ USE $\frac{(1+) - (1-)}{2\Delta x}$

$$\frac{u(x+\Delta x, t) - u(x-\Delta x, t)}{2\Delta x} = \frac{\partial u}{\partial x} + O(\Delta x^2)$$

CHOOSE $x = \pi/2 = x_N$ AND INTRODUCING THE GHOST MESH POINT $x_{N+1} = \pi/2 + \Delta x$

$$\text{WE HAVE } \frac{u_{N+1} - u_{N-1}}{2\Delta x} = 0$$

$$\text{OR } u_{N+1} = u_{N-1}$$

3. The motion of a damped string subject to an imposed load satisfies the following initial-boundary value problem:

$$u_{tt} + 2\gamma u_t = u_{xx} - 8 \sin x \cos x, \quad 0 < x < \pi, \quad t > 0 \quad (3)$$

$$u(0, t) = u(\pi, t) = 0 \quad (4)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin 3x.$$

- (a) Determine the static deflection $w(x)$ of the string (i.e., the steady solution), which is determined by solving (3) with $u_{tt} = u_t = 0$ and subject to the boundary conditions (4).

[5 marks]

- (b) Let $u(x, t) = w(x) + v(x, t)$ and determine the corresponding boundary value problem for $v(x, t)$.

[5 marks]

- (c) Assuming that $\gamma < 1$ use the method of separation of variables to solve for $v(x, t)$ and therefore $u(x, t)$.

[6 marks]

- (d) Now assuming no damping, i.e., letting $\gamma = 0$, use D'Alembert's solution (see the formula sheet) to determine $v(x, t)$ and therefore $u(x, t)$.

[4 marks]

[total 20 marks]

a) LET $w(x)$ BE THE STEADY SOLN

$$0 = w_{xx} - 8 \sin x \cos x \Rightarrow w_{xx} = 4 \sin 2x \Rightarrow w_x = -2 \cos 2x \quad \boxed{w(x) = -\sin 2x}$$

$$\text{NOTE } w(0) = 0 \quad \text{AND } w(\pi) = 0.$$

$$b) \cancel{w_{tt}} + v_{tt} + 2\gamma(\cancel{w_t} + v_t) = \{w_{xx} - 8 \sin x \cos x\} + \cancel{w_{xx}} \Rightarrow \boxed{v_{tt} + 2\gamma v_t = v_{xx}}$$

$$\text{BC: } 0 = u(0, t) = \cancel{w(0)} + v(0, t) \Rightarrow \boxed{v(0, t) = 0} \quad 0 = u(\pi, t) = \cancel{w(\pi)} + v(\pi, t) \Rightarrow \boxed{v(\pi, t) = 0}$$

$$\text{IC: } 0 = u(x, 0) = w(x) + v(x, 0) = -\sin 2x + v(x, 0) \Rightarrow \boxed{v(x, 0) = \sin 2x}$$

$$\sin 3x = u_t(x, 0) = \cancel{w_t} + v_t(x, 0) \Rightarrow \boxed{v_t(x, 0) = \sin 3x}$$

$$c) \text{ LET } v(x, t) = X(x)T(t) \Rightarrow X\ddot{T} + 2\gamma X\dot{T} = X''T \Rightarrow \frac{\ddot{T} + 2\gamma\dot{T}}{T(t)} = \frac{X''}{X} = -\lambda^2 \text{ const}$$

$$X] \quad X'' + \lambda^2 X = 0, \quad X(0) = 0 = X(\pi) \Rightarrow \lambda = \frac{n\pi}{\pi} = n, \quad n = 1, 2, \dots \quad X_n = \sin(nx)$$

$$T] \quad \ddot{T} + 2\gamma\dot{T} - n^2 T = 0 \quad \text{LET } T(t) = e^{\gamma t} \Rightarrow \gamma^2 + 2\gamma\gamma - n^2 = 0$$

$$\text{OR } \gamma = \left\{ -2\gamma \pm \sqrt{4\gamma^2 - 4n^2} \right\} / 2 = -\gamma \pm \sqrt{\gamma^2 - n^2} = -\gamma \pm i\sqrt{n^2 - \gamma^2} = -\gamma \pm i\mu_n \text{ since } \gamma < 1$$

$$\mu_n = \sqrt{n^2 - \gamma^2}$$

$$\therefore T_n(t) = e^{-\gamma t} [A_n \cos \mu_n t + B_n \sin \mu_n t]$$

$$\therefore v(x, t) = \sum_{n=1}^{\infty} e^{-\gamma t} [A_n \cos \mu_n t + B_n \sin \mu_n t] \sin(nx)$$

$$v_t(x, t) = \sum_{n=1}^{\infty} e^{-\gamma t} \left\{ (-\gamma) [A_n \cos \mu_n t + B_n \sin \mu_n t] + [A_n \mu_n \sin \mu_n t + B_n \mu_n \cos \mu_n t] \right\} \sin(nx)$$

$$\sin 2x = v(x, 0) = \sum_{n=1}^{\infty} A_n \sin nx \Rightarrow A_n = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases} = \delta_{n2}$$

$$\sin 3x = \sum_{n=1}^{\infty} [-\gamma [\delta_{n2}] + B_n \mu_n] \sin nx \Rightarrow B_2 = \gamma / \mu_2 \quad B_3 = \frac{1}{\mu_3} \quad B_n = 0 \quad n \neq 2, 3$$

$$\therefore u(x, t) = -\sin 2x + e^{-\gamma t} \left\{ \left[\cos \mu_2 t + \frac{\gamma}{\mu_2} \sin \mu_2 t \right] \sin(2x) + \frac{1}{\mu_3} \sin \mu_3 t \sin 3x \right\}$$

4. Consider the eigenvalue problem

$$x^2 y'' + xy' + \lambda y = 0 \quad (1)$$

$$y'(1) = 0 = y(e^{\pi/2}) \quad (2)$$

- (a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

- (b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the semi-annular region:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < e^{\pi/2}, \quad 0 < \theta < \pi$$

$$u(r, 0) = 0 \quad \text{and} \quad u(r, \pi) = f(r)$$

$$\frac{\partial u(1, \theta)}{\partial r} = 0 \quad \text{and} \quad u(e^{\pi/2}, \theta) = 0$$

[12 marks]
[total 20 marks]

$$a) F(x) = \frac{1}{x^2} \int \frac{x}{x^2} dx = \frac{1}{x^2} e^{\ln x} = 1/x.$$

$$(1) x^{\frac{1}{x}} \Rightarrow -xy'' + y' = \boxed{-(xy')' = \lambda y/x.} \quad \text{WHICH IS IN S-L FORM.}$$

NOW $x^2 y'' + xy' + \mu^2 y = 0$ (WHERE $\lambda = \mu^2$) IS A CAUCHY EULER EQ SO LET

$$y = x^{\gamma} \Rightarrow \gamma(\gamma-1) + \gamma + \mu^2 = 0 \quad \gamma = \pm i\mu.$$

$$\therefore y(x) = C_1 x^{i\mu} + C_2 x^{-i\mu} = A \cos \mu \ln x + B \sin \mu \ln x \quad \text{SINCE } x^{i\mu} = e^{i\mu \ln x}$$

$$y'(x) = -A \sin \mu \ln x \cdot (\mu/x) + B \cos \mu \ln x \cdot (\mu/x)$$

$$0 = y'(1) = -A \cdot 0 + B \cdot 1 \cdot \mu/1 \Rightarrow B = 0$$

$$0 = y(e^{\pi/2}) = A \cos \mu \ln e^{\pi/2} = A \cos(\mu \pi/2) \Rightarrow \mu_n = (2n+1) \quad n=1, 2, \dots$$

$$H_n(x) = \cos(2n+1) \ln x.$$

$$b) \text{ LET } u(r, \theta) = R(r) \Theta(\theta) \Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\left[\frac{r^2}{R\Theta} \right] \quad \frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = -\mu^2 \quad (-\text{SIGN BECAUSE HOMOGENEOUS BC IN } R)$$

$$\Theta] \quad \Theta'' - \mu^2 \Theta = 0 \quad \Theta(0) = 0 \Rightarrow \Theta = A \cosh \mu \theta + B \sinh \mu \theta$$

$$0 = \Theta(0) = A \Rightarrow A = 0$$

$$\Theta(\theta) = B \sinh \mu \theta.$$

$$R] \quad r^2 R'' + rR' - \mu^2 R = 0 \quad R(1) = 0 = R(e^{\pi/2}) \Rightarrow \mu_n = (2n+1) \quad R_n(r) = \cos(2n+1) \ln r$$

$$n=0, 1, \dots$$

$$\therefore \boxed{u(r, \theta) = \sum_{n=0}^{\infty} B_n \sinh \mu_n \theta \cos(\mu_n \ln r)}$$

$$f(r) = u(r, \pi) = \sum_{n=0}^{\infty} B_n \sinh(\mu_n \pi) \cos(\mu_n \ln r) = \sum_{n=0}^{\infty} b_n \cos(\mu_n \ln r)$$

$$\therefore \int_1^{e^{\pi/2}} \frac{f(r)}{r} \cos(\mu_m \ln r) dr = \sum_{n=0}^{\infty} b_n \int_1^{e^{\pi/2}} \cos(\mu_m \ln r) \cos(\mu_n \ln r) \frac{dr}{r}$$

$$\text{NOW } \int_1^{e^{\pi/2}} \cos(\mu_m \ln r) \cos(\mu_n \ln r) \frac{dr}{r} = \int_0^{\pi/2} \cos(\mu_m x) \cos(\mu_n x) dx = \delta_{mn} \frac{(\pi/2)}{2}$$

$$\therefore \boxed{B_n = \frac{b_n}{\sinh(\mu_n \pi)} = \frac{4}{\pi \sinh(\mu_n \pi)} \int_1^{e^{\pi/2}} \frac{f(r)}{r} \cos(\mu_n \ln r) dr}$$

$$\begin{aligned} x &= \ln r \\ dx &= dr/r \\ r=1 \Rightarrow x=0, \quad r=e^{\pi/2} \Rightarrow x=\pi/2 \end{aligned}$$

5. Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + 1 - xe^{-t}, \quad 0 < x < 1, t > 0 \\ u_x(0, t) &= e^{-t}, \text{ and } u(1, t) = t \\ u(x, 0) &= x. \end{aligned}$$

[20 marks]

FIRSTLY LOOK FOR A FUNCTION $W(x, t) = A(t)x + B(t)$ THAT SATISFIES THE NON-ZERO BC
 $e^{-t} = W_x = A(t); t = W(1) = e^{-t} \cdot 1 + B(t) \Rightarrow B(t) = t - e^{-t}$

$\therefore W(x, t) = e^{-t}x + t - e^{-t}$ SATISFIES THE BC.

NOW LET $u(x, t) = W(x, t) + v(x, t)$ AND DETERMINE THE BOUNDARY VALUE PROBLEM FOR v .

PDE: $u_t = W_t + v_t = -e^{-t}x + 1 + e^{-t} + v_t = \alpha^2(W_{xx} + v_{xx}) + 1 - xe^{-t} \Rightarrow v_t = \alpha^2 v_{xx} - e^{-t}$

BC: $e^{-t} = u_x(0, t) = W_x(0, t) + v_x(0, t) = e^{-t} + v_x(0, t) \Rightarrow v_x(0, t) = 0$

$t = u(1, t) = W(1, t) + v(1, t) = t + v(1, t) \Rightarrow v(1, t) = 0$

IC: $x = u(x, 0) = W(x, 0) + v(x, 0) = x - 1 + v(x, 0) \Rightarrow v(x, 0) = 1$

SINCE v IS SUBJECT TO HOMOGENEOUS BC WE CAN USE AN EIGENFUNCTION EXPANSION
 THE EIGENVALUES ASSOCIATED WITH THE EIGENVALUE PROBLEM $X'' + \lambda^2 X = 0$ $X'(0) = 0 = X(1)$
 ARE $\lambda_n = (2n+1)\pi/2$, $n=0, 1, 2, \dots$ AND THE CORRESPONDING EIGENFUNCTIONS ARE $X_n = \cos \lambda_n x$.

EXPAND THE SINK TERM $S(x, t) = -e^{-t} = \sum_{n=0}^{\infty} \hat{S}_n(t) \cos \lambda_n x$

$\therefore \hat{S}_n(t) = \frac{2}{1} \int_0^1 (-e^{-t}) \cos \lambda_n x dx = -2e^{-t} \frac{\sin(\lambda_n x)}{\lambda_n} \Big|_0^1 = -\frac{2}{\lambda_n} \sin\left(\frac{(2n+1)\pi}{2}\right) e^{-t} = \frac{(-1)^{n+1} \cdot 2}{\lambda_n} e^{-t} = C_n e^{-t}$

LET $v(x, t) = \sum_{n=0}^{\infty} \hat{V}_n(t) \cos \lambda_n x$ $\frac{\partial v}{\partial t} = \sum_{n=0}^{\infty} \frac{d\hat{V}_n}{dt} \cos \lambda_n x$ $\frac{\partial^2 v}{\partial x^2} = \sum_{n=0}^{\infty} \hat{V}_n(-\lambda_n^2) \cos \lambda_n x$

$\therefore 0 = v_t - \alpha^2 v_{xx} + e^{-t} = \sum_{n=0}^{\infty} \left\{ \frac{d\hat{V}_n}{dt} + \alpha^2 \lambda_n^2 \hat{V}_n - C_n e^{-t} \right\} \cos(\lambda_n x)$ SINCE $\cos \lambda_n x$ ARE LINEARLY INDEP. IT FOLLOWS THAT $\{ \} = 0$

$\therefore \frac{d\hat{V}_n}{dt} + \alpha^2 \lambda_n^2 \hat{V}_n = C_n e^{-t} \Rightarrow \frac{d}{dt} [e^{+\alpha^2 \lambda_n^2 t} \hat{V}_n] = C_n e^{(\alpha^2 \lambda_n^2 - 1)t}$

INTEGRATE $e^{+\alpha^2 \lambda_n^2 t} \hat{V}_n = \frac{C_n}{\alpha^2 \lambda_n^2 - 1} e^{(\alpha^2 \lambda_n^2 - 1)t} + d_n$

$\therefore \hat{V}_n(t) = \frac{C_n}{\alpha^2 \lambda_n^2 - 1} e^{-t} + d_n e^{-\alpha^2 \lambda_n^2 t}$

$\therefore v(x, t) = \sum_{n=0}^{\infty} \left[\frac{C_n e^{-t}}{\alpha^2 \lambda_n^2 - 1} + d_n e^{-\alpha^2 \lambda_n^2 t} \right] \cos \lambda_n x$

NOW $1 = v(x, 0) = \sum_{n=0}^{\infty} \left[\frac{C_n \cdot 1}{\alpha^2 \lambda_n^2 - 1} + d_n \right] \cos \lambda_n x = \sum_{n=0}^{\infty} a_n \cos \lambda_n x$

WHERE $a_n = \frac{2}{1} \int_0^1 1 \cos \lambda_n x dx = 2 \frac{\sin \lambda_n x}{\lambda_n} \Big|_0^1 = 2 \frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{\lambda_n} = -C_n$

$\therefore -C_n = \frac{C_n}{\alpha^2 \lambda_n^2 - 1} + d_n \Rightarrow d_n = -C_n \left[\frac{\alpha^2 \lambda_n^2 - 1 + 1}{\alpha^2 \lambda_n^2 - 1} \right] = -\frac{\alpha^2 \lambda_n^2 C_n}{\alpha^2 \lambda_n^2 - 1}$

$\therefore u(x, t) = e^{-t}(x-1) + t + \sum_{n=0}^{\infty} \frac{C_n}{\alpha^2 \lambda_n^2 - 1} [e^{-t} - \alpha^2 \lambda_n^2 e^{-\alpha^2 \lambda_n^2 t}] \cos(\lambda_n x)$

WHERE $C_n = \frac{(-1)^{n+1} \cdot 2}{\lambda_n}$ AND $\lambda_n = \frac{(2n+1)\pi}{2}$ $n=0, 1, \dots$