Math 1107 Practice Final Exam

Disclaimer: This practice exam does NOT intend to cover all topics that may appear on the final exam. Any topic covered during the course may be tested on the final. Solving only these problems does not constitute sufficient preparation for the final.

Multiple choice problems

1. Which of the following linear transformations is 1-1? (only one is 1-1)

(a) $T(x, y) = (x + y, x + y)$

(b) $T(x, y) = 2x + 3y$

(c) $T(x, y, z) = (2x - 3y, x - 4y)$

(d) $T(x, y, z) = (-y, 3x, 8z)$

(e) $T(x, y, z) = (x + y, y + z, x + 2y + z)$

2. Let P be the plane passing through the point $(1, 1, 0)$ with normal vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Which of the following points is in P?

(a) $(-1, 1, 1)$

(b) $(-1, -2, 0)$

(c) $(-1, 1, 3)$

(d) $(3, 4, -1)$

(e) $(1, -2, 1)$

3a. Which of the following is NOT a subspace of $\mathbb{R}^3$?

(a) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(b) The line through $(4, 2, -6)$ in the direction $u = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

(c) The plane $x + 2y - 5z = 0$.

(d) The span of $u = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix}$.

(e) The set of solutions to the linear system $Ax = 0$ where $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -4 \\ -2 & 4 & 10 \end{pmatrix}$.

3b. Which one of the following matrices is invertible? (Only one is invertible)

(a) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & 1 & -3 \\ 0 & 0 & 8 \\ 0 & 0 & -4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -6 & -3 \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & -1 \\ 4 & 1 & -7 \end{pmatrix}$

4. Let A be a 4×4 matrix such that the system of equations $Ax = 0$ has infinitely many solutions. Which of the following statements is true? (Only one is true)

(a) The rows of A span $\mathbb{R}^4$.

(b) The matrix A is invertible.

(c) The rank of A is 4.

(d) The rows of A are linearly dependent.

(e) The kernel of the linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^4$ has dimension 0.

5. Suppose that A and B are 4×4 matrices and that $\det(A) = \det(B) = -\frac{1}{2}$. Then $\det\left(-A^3BA^T(-3B^2)(-A)^{-1}\right)$ is

(a) $-\frac{81}{64}$

(b) $\frac{81}{64}$

(c) $\frac{3}{64}$

(d) $-\frac{3}{64}$

6. Which of the following vectors is orthogonal to both $\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$?

(a) $\begin{pmatrix} 6 \\ 16 \\ 6 \end{pmatrix}$

(b) $\begin{pmatrix} 3 \\ 8 \\ -3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 \\ -8 \\ 3 \end{pmatrix}$

(d) $\begin{pmatrix} -3 \\ 8 \\ 3 \end{pmatrix}$

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that the kernel of T is a plane (through the origin). Which of the following describes the image of T?

(a) The image of T is the zero subspace $\{(0,0,0)\}$.

(b) The image of T is a line through the origin.

(c) The image of T is a plane through the origin.

(d) The image of T is $\mathbb{R}^3$

(e) There is not enough information to describe the image of T.

8. Let A be an invertible $n \times n$ matrix, B a $p \times s$ matrix, and C a $q \times t$ matrix. If the matrix expression $AB^T - (CA^{-1})^T$ is defined, then only one of the following is a necessary restriction on the sizes n, p, s, q, t. Which one?

(a) $n = p = q = s = t$

(b) $n = s = q$ and $p = t$

(c) $n = s = t$ and $p = q$

(d) $n = s = p$ and $q = t$

9. For what value(s) of k are the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix}$ linearly independent?

(a) $k = 0$

(b) $k = 2$

(c) All $k \neq 2$.

(d) All $k \neq 0$.

(e) No value of k.

10. If $z = 3 - 4i$, what is $\frac{1}{z}$ in standard form?

(a) $\frac{3}{25} + \frac{4}{25}i$

(b) $3 + 4i$

(c) $-3 + 4i$

(d) $-\frac{3}{7} - \frac{4}{7}i$

(e) $\frac{1}{3} - \frac{1}{4}i$

11. Which of the following is an eigenvector of the matrix $\begin{pmatrix} 0 & -2 & 4 \\ -3 & -1 & 8 \\ -2 & -2 & 7 \end{pmatrix}$?

(a) $\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

(d) $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

(e) $\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$

12. Let $S = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$. Which of the following sets is a basis of S?

(a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}$

(e) $\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right\}$

13. Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 0 \\ 2 \\ -2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $v_4 = \begin{pmatrix} 1 \\ -1 \\ -3 \\ 1 \end{pmatrix}$, $v_5 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, and $S = \text{span}\{v_1, v_2, v_3, v_4, v_5\}$. What is the dimension of S?

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

14. Let $A = \begin{pmatrix} 2i & -2+i \\ 1 & 1+i \end{pmatrix}$. Which of the following is A^{-1}?

(a) $\begin{pmatrix} -1+i & 1+2i \\ -i & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 1-i & 1+2i \\ -i & 2 \end{pmatrix}$

(c) $\begin{pmatrix} -1+i & i \\ -1-2i & -2 \end{pmatrix}$

(d) $\begin{pmatrix} 1+i & 2-i \\ -1 & 2i \end{pmatrix}$

(e) $\begin{pmatrix} 1-i & -1-2i \\ i & 2 \end{pmatrix}$

15. For what value(s) of k is the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$ singular (non-invertible)?

(a) All real numbers.

(b) All real numbers except 0 and 1.

(c) All real numbers except 1 and -1.

(d) $k = 0, 1$

(e) $k = 1, -1$

Long answer problems

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_3, x_2 - 4x_4, x_1 + x_3 - x_4, x_2 - 3x_3 + x_4).$$

(a) Find the standard matrix of T.

(b) Find a basis for the image of T.

(c) Find a basis for the kernel (nullspace) of T.

2. Let $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$.

(a) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

(b) Compute A^{2011}. (Hint: Use part (a))

3. Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{pmatrix}.$$

(a) Solve the homogeneous linear system $Ax = 0$.

(b) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by $T(v) = Av$. Find a basis for $\ker(T)$.

(c) Is T 1-1?

(d) Is T onto?

4. Let $u = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$. Find the unique vector $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ in $\mathbb{R}^3$ such that $u \cdot v = 1$, v is orthogonal to w, $||v|| = \sqrt{3}$, and $v_2 > 0$.

Solutions

Multiple choice problems

1. (d)
2. (d)
3a. (b)
3b. (a)
4. (d)
5. (b)
6. (a)
7. (b)
8. (c)
9. (c)
10. (a)
11. (e)
12. (b)
13. (c)
14. (e)
15. (d)

Long answer problems

1. Compute $T(e_1)$, $T(e_2)$, $T(e_3)$, and $T(e_4)$ and place these vectors into columns of a matrix. So the standard matrix is

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -4 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1 \end{pmatrix}.$$

We find that the row-reduced echelon form of A is the identity matrix I, so the four columns of A form a basis of $\text{Im}(T)$ and the kernel of T is the trivial subspace

$$\ker(T) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

which has no basis.

In fact, since A is invertible, the image of T (which is the same of the column space of A) is all of $\mathbb{R}^4$ and so any basis of $\mathbb{R}^4$ is a basis of the image of T.

2. For (a), we find that A has eigenvalue $\lambda = 3$ with corresponding eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and eigenvalue $\lambda = 2$ with eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. So

$$P = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

or

$$P = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

In either case, we may rescale each column of P by any non-zero scalar.

For (b), we have

$$\begin{aligned} A^{2011} &= (PDP^{-1})^{2011} = PD^{2011}P^{-1} \\ &= \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^{2011} & 0 \\ 0 & 2^{2011} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (-2)3^{2011} & -2^{2011} \\ 3^{2011} & 2^{2011} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (2)3^{2011} - 2^{2011} & (2)3^{2011} - 2^{2012} \\ -3^{2011} + 2^{2011} & -3^{2011} + 2^{2012} \end{pmatrix} \end{aligned}$$

3. (a) The matrix A row-reduces to

$$R = \begin{pmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, variables x_3 and x_4 are non-leading so setting $x_3 = s$, $x_4 = t$ we obtain

$$\begin{aligned} x_1 &= -4s + 4t \\ x_2 &= 5s - 8t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

and so the general solution to $Ax = 0$ is

$$x = s \begin{pmatrix} -4 \\ 5 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ -8 \\ 0 \\ 1 \end{pmatrix}.$$

(b) The kernel of T is exactly the set of solutions to $Ax = 0$, so the vectors $(-4, 5, 1, 0)$ and $(4, -8, 0, 1)$, which are linearly independent, form a basis of $\ker(T)$.

(c) Since the kernel of T is NOT equal to $\{(0, 0, 0)\}$, T is not 1-1.

(d) From the Rank Theorem, we know that

$$\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = 4$$

and from (c) we know that $\dim(\ker(T)) = 2$. Hence $\dim(\mathrm{Im}(T)) = 2$ and so the image of T is NOT $\mathbb{R}^3$ so T is not onto. Another way to see that $\dim(\mathrm{Im}(T)) = 2$ is that $\dim(\mathrm{Im}(T))$ is always equal to the rank of A, which is 2 since R has two non-zero rows.

4. $v = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$