CARLETON UNIVERSITY

FINAL
EXAMINATION
December 2010

DURATION: 3 HOURS

School of Mathematics and Statistics — Math 1107 D,E

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AUTHORIZED MEMORANDA

NON-PROGRAMMABLE CALCULATORS ALLOWED

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Family Name: ____________________ **First (Given) Name:** ________________

Student Number: _______________

Instructions:

1. This is a closed book exam.
2. Answer **all** questions and show **all** appropriate steps in your work; otherwise only partial marks may be awarded.

Problem	Maximum mark	Actual mark
1-15	30	
16	10	
17	10	
18	12	
19	10	
20	10	
21	18	
Total	100	

PART I: Multiple choice questions (2 marks each)

Please CIRCLE your answer on the answer sheet on page 6)

No partial marks here. No justification is required.

1. Let $A = \left[\begin{array}{cccc} 1 & 2 & 5 & | & 1 \\ 0 & 7 & -2 & | & 2 \\ 0 & 0 & (h-1)(h+1) & | & (h-1)(h-2) \end{array}\right]$ be the augmented matrix of a linear system. For which value of h is the system INCONSISTENT?

   (a) $h = 1$

   (b) $h = 2$

   (c) $h = -1$

   (d) $h = -2$

2. Let A be a 4×4 matrix row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-1 & y \\ 0 & 0 & 0 & x \end{bmatrix}$. For which values of x and y is A invertible?

   (a) $x = 1$ and $y = 1$

   (b) $x = 2$ and $y = 1$

   (c) $x = 0$ and $y = 0$

   (d) $x \neq 1$, $x \neq 2$, $x \neq 0$ and $y = 0$

3. Let $T \colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Suppose $T(u) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $T(v) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. What is $T(2u - 3v)$?

   (a) $\begin{bmatrix} 3 \\ 17 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ -17 \end{bmatrix}$ (c) $\begin{bmatrix} 15 \\ 13 \end{bmatrix}$ (d) $\begin{bmatrix} -15 \\ -13 \end{bmatrix}$

4. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 4 & 1 & 2 \end{bmatrix}$. What is $E_1^2 E_2^3 E_3^4 A$?

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 4 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & -2 \\ 4 & 1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 2 & 2 \\ 4 & 1 & 2 \end{bmatrix}$

5. Which of the following matrices is **NOT** in **reduced row echelon form**?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6. Let A, B and C be 4×4 matrices such that $\det A = 4$, $\det B = -3$, and $\det C = 2$. Then $\det(2AB^TC^{-1})$ is:

(a) -48 (b) -96 (c) -12 (d) 96

7. If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5$ then $\det \begin{bmatrix} d & e & f \\ 2a & 2b & 2c \\ g-a & h-b & i-c \end{bmatrix}$ is:

(a) -20 (b) -10 (c) 10 (d) 20

8. The adjugate of $A = \begin{bmatrix} 1 & -6 \\ 7 & 4 \end{bmatrix}$ is:

(a) $\begin{bmatrix} 4 & 6 \\ -7 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 4 \\ 1 & -7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 7 \\ -6 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 6 \\ -7 & -4 \end{bmatrix}$

9. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 6 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

If the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$ is $[x]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, what is the value of c?

(a) -3 (b) 3 (c) -8 (d) 4

10. Which of the following statements is **FALSE**?

(a) Each elementary matrix is invertible.

(b) Let $\mathbf{0}$ denote the zero vector in $\mathbb{R}^3$. Then both $\mathbb{R}^3$ and $\{\mathbf{0}\}$ are subspace of $\mathbb{R}^3$.

(c) Let A, B, and C be square matrices of the same size . If $AB = AC$, then $B = C$.

(d) Any set of vectors in $\mathbb{R}^n$ containing the zero vector $\mathbf{0}$ is linearly dependent.

11. What is the complex conjugate of $z = 2 - \sqrt{3}i$?

(a) $-2 - \sqrt{3}i$ (b) $-2 + \sqrt{3}i$ (c) $\sqrt{3} - 2i$ (d) $2 + \sqrt{3}i$

12. Which of the following 3 matrices have columns which form an orthogonal basis of $\mathbb{R}^3$?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) only B (b) only A and B (c) only A and C (d) $A, B,$ and C

13. Write $z = \dfrac{2+i}{1-i}$ in the form $a + bi$ with a and b real numbers by rationalizing the denominator.

(a) $\frac{1}{2} + \frac{3}{2}i$ (b) $\frac{2-i}{1+i}$ (c) $\frac{3}{2} + \frac{3}{2}i$ (d) $\frac{1}{2} - \frac{3}{2}i$

14. Which of the following sets is an **orthonormal basis** for $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\}$,

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$,

(d) $\left\{ \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$

15. What is the dimension of the subspace

$$U = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} \right\} \text{ of } \mathbb{R}^3?$$

(a) 1 (b) 2 (c) 3 (d) 5

Answer Sheet:

(1)	A	B	C	D
(2)	A	B	C	D
(3)	A	B	C	D
(4)	A	B	C	D
(5)	A	B	C	D
(6)	A	B	C	D
(7)	A	B	C	D
(8)	A	B	C	D
(9)	A	B	C	D
(10)	A	B	C	D
(11)	A	B	C	D
(12)	A	B	C	D
(13)	A	B	C	D
(14)	A	B	C	D
(15)	A	B	C	D

Solutions:

$1 \to 5$	C	D	B	D	C
$6 \to 10$	B	B	A	B	C
$11 \to 15$	D	B	A	D	B

PART II: Long answer questions

Show all your work.

16. (a) [5 marks] Find all solutions to the following system. Write the solutions in parametric vector form.

$$\begin{cases} x + 3y + z = 1 \\ -4x - 9y + 2z = -1 \\ -3y - 6z = -3 \end{cases}$$

(b) [5 marks] Use Cramer's rule to compute the solutions of the following system:

$$\begin{cases} 2x + y + z = 3 \\ x - y - z = 0 \\ x + 2y + z = 0 \end{cases}$$

Solution: (a) The reduced echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

(b)

$$x = 1, \ y = -2, \ z = 3$$

(*Additional page for Question 16.*)

17. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ x_2 + 2x_3 + 3x_4 \\ x_3 + 2x_4 \end{bmatrix}.$$

(a) [3 marks] Find the standard matrix of T.

(b) [3 marks] Find the image of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(c) [4 marks] Is the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the range of T? Explain.

Solution: (a) The standard matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 9 & 0 & 1 & 2 \end{bmatrix}$$

(b)

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 30 \\ 20 \\ 11 \end{bmatrix}$$

(c) Yes. Since the row rank of the standard matrix is 3.

18. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -4 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) [8 marks] Find the inverse of A.

(b) [4 marks] Solve $A\mathbf{x} = \mathbf{b}$.

Solution: (a) The inverse matrix is

$$\begin{bmatrix} -4 & 4 & 3 \\ 1 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

(b)

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

19. Let

$$A = \begin{bmatrix} 1 & 2 & -5 & 0 \\ 5 & 4 & 5 & 2 \\ -2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 2 \end{bmatrix}.$$

(a) [8 marks] Find the determinant of A.

(b) [2 marks] Conclude whether or not A is invertible. Explain.

Solution: (a)

$$|A| = -62$$

(b) A is invertible , since $|A| \neq 0$.

20. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) [1.5 marks] Find the characteristic polynomial of A.

(b) [1 marks] Find all the eigenvalues of A.

(c) [4.5 marks] Find a basis for each eigenspace of A.

(d) [3 marks] If possible, find an invertible matrix P and a

diagonal matrix D such that $P^{-1}AP = D$.

Solution: (a)

$$C_A(\lambda) = (1-\lambda)(2-\lambda)(3-\lambda)$$

(b)

$$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$$

(c)

$$\text{For } \lambda_1 = 1,\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\quad \text{For } \lambda_2 = 2,\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\quad \text{For } \lambda_3 = 3,\ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

(d)

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix},\quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

21. Consider the matrix $U = \begin{bmatrix} 1 & 3 & 2 & 0 & 4 \\ 1 & 0 & -1 & -3 & 1 \\ -1 & 3 & 4 & 5 & 1 \end{bmatrix}$.

(a) [5 marks] Give a basis of Col(U) (the column space of U).

(b) [5 marks] Give a basis of Nul(U) (the null space of U).

(c) [2 marks] Determine rank(U) and dimNul(U).

(d) [4 marks] Does any basis of col(U) also form a basis for $\mathbb{R}^3$? Explain your answer.

(e) [2 marks] Are the three middle columns of U linearly **independent** or linearly **dependent**?

Solution: The RREF of U is

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} \right\}$$

(b)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(c) Rank (U)=3, Dim [Nul(U)]=2

(d) Yes, since rank is 3.

(e) Independent.