COMP 2804 — Solutions Assignment 3

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: James Bond
- Student number: 007

Question 2: Consider two dice, each one having one face showing the letter a, two faces showing the letter b, and the remaining three faces showing the letter c. You roll each die once, independently of the other die.

- What is the sample space?
- Define the events

  $$A = \text{"at least one of the two dice shows the letter } b \text{ on its top face"}$$

  and

  $$B = \text{"both dice show the same letter on their top faces".}$$

  Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(A \mid B)$.

Solution: The sample space is the set S of all possible outcomes:

$$S = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}.$$

If we roll one die, we have

$$\begin{aligned} \Pr(a) &= 1/6 \\ \Pr(b) &= 2/6 = 1/3 \\ \Pr(c) &= 3/6 = 1/2 \end{aligned}$$

If we roll both dies, independently of each other, we have

$$\begin{aligned} \Pr(a,a) &= 1/6 \cdot 1/6 = 1/36 \\ \Pr(a,b) &= 1/6 \cdot 1/3 = 1/18 \\ \Pr(a,c) &= 1/6 \cdot 1/2 = 1/12 \\ \Pr(b,a) &= 1/3 \cdot 1/6 = 1/18 \\ \Pr(b,b) &= 1/3 \cdot 1/3 = 1/9 \\ \Pr(b,c) &= 1/3 \cdot 1/2 = 1/6 \\ \Pr(c,a) &= 1/2 \cdot 1/6 = 1/12 \\ \Pr(c,b) &= 1/2 \cdot 1/3 = 1/6 \\ \Pr(c,c) &= 1/2 \cdot 1/2 = 1/4 \end{aligned}$$

If we write the events A and B as subsets of the sample space, we get

$$A = \{(a,b),(b,a),(b,b),(b,c),(c,b)\}$$

and

$$B = \{(a,a),(b,b),(c,c)\}.$$

It follows that

$$\begin{aligned}\Pr(A) &= \Pr(a,b)+\Pr(b,a)+\Pr(b,b)+\Pr(b,c)+\Pr(c,b)\\ &= 1/18+1/18+1/9+1/6+1/6\\ &= 5/9\end{aligned}$$

and

$$\begin{aligned}\Pr(B) &= \Pr(a,a)+\Pr(b,b)+\Pr(c,c)\\ &= 1/36+1/9+1/4\\ &= 7/18.\end{aligned}$$

By definition,

$$\Pr(A \mid B) = \frac{\Pr(A\cap B)}{\Pr(B)}.$$

Since

$$A \cap B = \{(b,b)\},$$

we have

$$\Pr(A\cap B) = \Pr(b,b) = 1/9.$$

We conclude that

$$\Pr(A \mid B) = \frac{1/9}{7/18} = 2/7.$$

Question 3: A group of ten people sits down, uniformly at random, around a table. Lindsay and Simon are part of this group. Determine the probability that Lindsay and Simon sit next to each other.

Solution: We number the positions around the table in clockwise order from 1 up to 10.

We first observe that there are 10! ways for the ten people to sit down.

We will use (p,q) to denote that Lindsay sits at position p and Simon sits at position q. There are 20 ways for Lindsay and Simon to sit next to each other:

$$(1,2),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8),(8,9),(9,10),(10,1)$$

and

$$(2,1),(3,2),(4,3),(5,4),(6,5),(7,6),(8,7),(9,8),(10,9),(1,10).$$

For each of these 20 possibilities, there are 8! ways for the other people to sit down. Therefore, the probability that Lindsay and Simon sit next to each other is equal to

$$\frac{20 \cdot 8!}{10!} = \frac{20}{9 \cdot 10} = \frac{2}{9}.$$

Question 4: You flip a fair coin, independently, three times. Define the events

$$A = \text{“the first flip results in heads”},$$

and

$$B = \text{“the coin comes up heads exactly once”}.$$

Determine $\Pr(A \mid B)$ and $\Pr(B \mid A)$.

Solution: The sample space is the set

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\},$$

which is of size 8. The events A and B correspond to the subsets

$$A = \{HHH, HHT, HTH, HTT\}$$

and

$$B = \{HTT, THT, TTH\}.$$

Since

$$A \cap B = B \cap A = \{HTT\},$$

we get

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/8}{3/8} = 1/3$$

and

$$\Pr(B \mid A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{1/8}{4/8} = 1/4.$$

Question 5:

- Consider three events A, B, and C in a sample space S, and assume that $\Pr(B \cap C) \neq 0$ and $\Pr(C) \neq 0$. Prove that

$$\Pr(A \cap B \cap C) = \Pr(A \mid B \cap C) \cdot \Pr(B \mid C) \cdot \Pr(C).$$

- You have a fair die and do the following experiment:
  - Roll the die once; let x be the outcome.

– Roll the die x times (independently); let y be the smallest outcome of these x rolls.

– Roll the die y times (independently); let z be the largest outcome of these y rolls.

Determine

$$\Pr(x = 1 \text{ and } y = 2 \text{ and } z = 3).$$

Solution: Using the definition of conditional probability, we get

$$\begin{aligned}\Pr(A \mid B \cap C) \cdot \Pr(B \mid C) \cdot \Pr(C) &= \frac{\Pr(A \cap B \cap C)}{\Pr(B \cap C)} \cdot \frac{\Pr(B \cap C)}{\Pr(C)} \cdot \Pr(C) \\ &= \Pr(A \cap B \cap C),\end{aligned}$$

proving the first part.

To prove the second part, we define the events

$$A = \text{“}z = 3\text{”},$$

$$B = \text{“}y = 2\text{”},$$

and

$$C = \text{“}x = 1\text{”}.$$

To determine $\Pr(C)$: We roll the die once; event C occurs if and only if this roll results in 1. Thus,

$$\Pr(C) = 1/6.$$

To determine $\Pr(B \mid C)$: We are given that event C occurs, i.e., $x = 1$. Thus, when the value of y is determined, we roll the die once. Event B occurs if and only if this roll results in 2. Therefore,

$$\Pr(B \mid C) = 1/6.$$

To determine $\Pr(A \mid B \cap C)$: We are given that both events B and C occur, i.e., $x = 1$ and $y = 2$. Thus, when the value of z is determined, we roll the die twice. Event A occurs if and only if the largest outcome of these two rolls is 3. The outcomes for this to happen are

$$(3, 1), (3, 2), (3, 3), (1, 3), (2, 3),$$

out of the total number of 36 outcomes when rolling a die twice. Therefore,

$$\Pr(A \mid B \cap C) = 5/36.$$

We conclude that

$$\begin{aligned}\Pr(x = 1 \text{ and } y = 2 \text{ and } z = 3) &= \Pr(A \cap B \cap C) \\ &= \Pr(A \mid B \cap C) \cdot \Pr(B \mid C) \cdot \Pr(C) \\ &= 5/36 \cdot 1/6 \cdot 1/6 \\ &= 5/1296.\end{aligned}$$

Question 6: Consider two independent events A and B in a sample space S. Assume that A and B are disjoint, i.e., $A \cap B = \emptyset$. What can you say about $\Pr(A)$ and $\Pr(B)$? Justify your answer.

Solution: Since A and B are independent, we have

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

Since A and B are disjoint, we have

$$\Pr(A \cap B) = \Pr(\emptyset) = 0.$$

It follows that

$$\Pr(A) \cdot \Pr(B) = 0,$$

implying that at least one of $\Pr(A)$ and $\Pr(B)$ is equal to zero.

Question 7: Annie, Boris, and Charlie write an exam that consists of only one question: *What is* 26 *times* 26*?* Calculators are not allowed during the exam. Both Annie and Boris are pretty clever and each of them gives the correct answer with probability 9/10. Charlie has trouble with two-digit numbers and gives the correct answer with probability 6/10.

- Assume that the three students do not cheat, i.e., each student answers the question independently of the other two students. Determine the probability that at least two of them give the correct answer.
- Assume that Annie and Boris do not cheat, but Charlie copies Annie's answer. Determine the probability that at least two of them give the correct answer.

Hint: The answer to the second part is smaller than the answer to the first part.

Solution: We define the events

$$A = \text{"Annie gives the correct answer"},$$

$$B = \text{"Boris gives the correct answer"},$$

and

$$C = \text{"Charlie gives the correct answer"}.$$

We are given that

$$\Pr(A) = \Pr(B) = 9/10$$

and

$$\Pr(C) = 6/10 = 3/5.$$

First we consider the case when the three students do not cheat. Here are the possibilities for the event that at least two students give the correct answer:

- $A \cap B \cap C$: This happens with probability

$$9/10 \cdot 9/10 \cdot 3/5 = 243/500.$$

- $A \cap B \cap \overline{C}$: This happens with probability

$$9/10 \cdot 9/10 \cdot 2/5 = 162/500.$$

- $A \cap \overline{B} \cap C$: This happens with probability

$$9/10 \cdot 1/10 \cdot 3/5 = 27/500.$$

- $\overline{A} \cap B \cap C$: This happens with probability

$$1/10 \cdot 9/10 \cdot 3/5 = 27/500.$$

Therefore, the probability that at least two of the students give the correct answer is equal to

$$243/500 + 162/500 + 27/500 + 27/500 = 459/500 = 0.918.$$

Next, we consider the case when Charlie copies Annie's answer. In this case, at least two of the students give the correct answer if and only if Annie gives the correct answer. Therefore, the probability that at least two of the students give the correct answer is equal to

$$\Pr(A) = 9/10 = 0.9.$$

Question 8: For $i \in \{1, 2\}$, consider the game G_i, in which two players P_1 and P_2 take turns flipping, independently, a fair coin, where P_i starts. The game ends as soon as heads comes up. The player who flips heads first is the winner of the game G_i. For $j \in \{1, 2\}$, define the event

$$B_{ij} = \text{“}P_j \text{ wins the game } G_i\text{”}.$$

In class (see Section 5.14.2), we have seen that

$$\Pr(B_{11}) = \Pr(B_{22}) = 2/3 \tag{1}$$

and

$$\Pr(B_{12}) = \Pr(B_{21}) = 1/3. \tag{2}$$

Consider the game G, in which P_1 and P_2 take turns flipping, independently, a fair coin, where P_1 starts. The game ends as soon as a second heads comes up. The player who flips the second heads wins the game. Define the event

$$A = \text{“}P_1 \text{ wins the game } G\text{”}.$$

In class (see Section 5.14.3), we used an infinite series to show that

$$\Pr(A) = 4/9. \tag{3}$$

Use the Law of Total Probability to give an alternative proof of (3). You are allowed to use (1) and (2).

Solution: Consider the events B_{11} and B_{12}. We observe that

- $\Pr(B_{11}) = 2/3 > 0$,
- $\Pr(B_{12}) = 1/3 > 0$,
- exactly one of B_{11} and B_{12} is guaranteed to occur.

Therefore, we can apply the Law of Total Probability and get

$$\Pr(A) = \Pr(A \mid B_{11}) \cdot \Pr(B_{11}) + \Pr(A \mid B_{12}) \cdot \Pr(B_{12}).$$

We know that $\Pr(B_{11}) = 2/3$ and $\Pr(B_{12}) = 1/3$. So it remains to determine $\Pr(A \mid B_{11})$ and $\Pr(A \mid B_{12})$.

- We start with $\Pr(A \mid B_{11})$:
  - A is the event that P_1 starts and P_1 flips the second heads.
  - We are given that the event B_{11} occurs; thus, P_1 flips the first heads.
  - The bitflips after the first heads can be seen as "P_2 starts and P_1 flips the first heads"; this is the same as the event B_{21}.
  - Therefore, $\Pr(A \mid B_{11}) = \Pr(B_{21}) = 1/3$.
- Next we determine $\Pr(A \mid B_{12})$:
  - A is the event that P_1 starts and P_1 flips the second heads.
  - We are given that the event B_{12} occurs; thus, P_2 flips the first heads.
  - The bitflips after the first heads can be seen as "P_1 starts and P_1 flips the first heads"; this is the same as the event B_{11}.
  - Therefore, $\Pr(A \mid B_{12}) = \Pr(B_{11}) = 2/3$.

We conclude that

$$\begin{aligned} \Pr(A) &= \Pr(A \mid B_{11}) \cdot \Pr(B_{11}) + \Pr(A \mid B_{12}) \cdot \Pr(B_{12}) \\ &= 1/3 \cdot 2/3 + 2/3 \cdot 1/3 \\ &= 4/9. \end{aligned}$$

Question 9: You would like to generate a uniformly random bit, i.e., with probability 1/2, this bit is 0, and with probability 1/2, it is 1. You find a coin in your pocket, but you are not sure if it is a fair coin: It comes up heads (H) with probability p and tails (T) with probability $1-p$, for some real number p that is unknown to you. In particular, you do not know if $p = 1/2$. In this question, you will show that this coin can nevertheless be used to generate a uniformly random bit.

Consider the following recursive algorithm GETRANDOMBIT, which does not take any input:

Algorithm GETRANDOMBIT**:**

```
// all coin flips made are mutually independent
flip the coin twice;
if the result is HT
then return 0
else if the result is TH
      then return 1
      else GetRandomBit
      endif
endif
```

- The sample space S is the set of all sequences of coin flips that can occur when running algorithm GETRANDOMBIT. Determine this sample space S.
- Prove that algorithm GETRANDOMBIT returns a uniformly random bit.

Solution: The algorithm repeatedly flips the coin twice. It terminates (and returns 0 or 1) as soon as it flips HT or TH. In other words, if it flips HH or TT, the algorithm repeats itself. The sample space S is therefore the union of the two sets

$$A = \{b_1 b_2 \dots b_{2n} HT : n \geq 0, b_1 = b_2, b_3 = b_4, \dots, b_{2n-1} = b_{2n}\}$$

and

$$B = \{b_1 b_2 \dots b_{2n} TH : n \geq 0, b_1 = b_2, b_3 = b_4, \dots, b_{2n-1} = b_{2n}\}.$$

For any bitstring in A, the algorithm returns 0, and for any bitstring in B, the algorithm returns 1. In other words, A is the event "the algorithm returns the bit 0".

Consider a bitstring

$$b_1 b_2 \dots b_{2n} HT$$

in A. Thus, $n \geq 0$ and $b_1 = b_2, b_3 = b_4, \dots, b_{2n-1} = b_{2n}$. In two flips of the coin, there are two ways to flip the same bit: either both flips are H, which happens with probability p^2, or both flips are T, which happens with probability $(1-p)^2$. Therefore, the probability that two flips give the same result is equal to $p^2 + (1-p)^2$. It follows that the probability of flipping the bitstring $b_1 b_2 \dots b_{2n} HT$ (which we assume to be in the set A) is equal to

$$\left(p^2 + (1-p)^2\right)^n p(1-p).$$

We conclude that

$$\begin{aligned}
\Pr(\text{ the algorithm returns the bit 0 }) &= \Pr(A) \\
&= \sum_{n=0}^{\infty} \left(p^2 + (1-p)^2\right)^n p(1-p) \\
&= p(1-p) \sum_{n=0}^{\infty} \left(p^2 + (1-p)^2\right)^n \\
&= p(1-p) \cdot \frac{1}{1 - (p^2 + (1-p)^2)} \\
&= \frac{p(1-p)}{2p - 2p^2} \\
&= \frac{p(1-p)}{2p(1-p)} \\
&= \frac{1}{2}.
\end{aligned}$$