COMP 2804 — Assignment 3

Due: November 19, before 23:59 pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: Consider two dice, each one having one face showing the letter a, two faces showing the letter b, and the remaining three faces showing the letter c. You roll each die once, independently of the other die.

- What is the sample space?
- Define the events

  $$A = \text{“at least one of the two dice shows the letter } b \text{ on its top face”}$$

  and

  $$B = \text{“both dice show the same letter on their top faces”}.$$

  Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(A \mid B)$.

Question 3: A group of ten people sits down, uniformly at random, around a table. Lindsay and Simon are part of this group. Determine the probability that Lindsay and Simon sit next to each other.

Question 4: You flip a fair coin, independently, three times. Define the events

$$A = \text{“the first flip results in heads”},$$

and

$$B = \text{“the coin comes up heads exactly once”}.$$

Determine $\Pr(A \mid B)$ and $\Pr(B \mid A)$.

Question 5:

- Consider three events A, B, and C in a sample space S, and assume that $\Pr(B \cap C) \neq 0$ and $\Pr(C) \neq 0$. Prove that

$$\Pr(A \cap B \cap C) = \Pr(A \mid B \cap C) \cdot \Pr(B \mid C) \cdot \Pr(C).$$

- You have a fair die and do the following experiment:
  - Roll the die once; let x be the outcome.
  - Roll the die x times (independently); let y be the smallest outcome of these x rolls.
  - Roll the die y times (independently); let z be the largest outcome of these y rolls.

  Determine

$$\Pr(x = 1 \text{ and } y = 2 \text{ and } z = 3).$$

Question 6: Consider two independent events A and B in a sample space S. Assume that A and B are disjoint, i.e., $A \cap B = \emptyset$. What can you say about $\Pr(A)$ and $\Pr(B)$? Justify your answer.

Question 7: Annie, Boris, and Charlie write an exam that consists of only one question: *What is* 26 *times* 26*?* Calculators are not allowed during the exam. Both Annie and Boris are pretty clever and each of them gives the correct answer with probability 9/10. Charlie has trouble with two-digit numbers and gives the correct answer with probability 6/10.

- Assume that the three students do not cheat, i.e., each student answers the question independently of the other two students. Determine the probability that at least two of them give the correct answer.
- Assume that Annie and Boris do not cheat, but Charlie copies Annie's answer. Determine the probability that at least two of them give the correct answer.

Hint: The answer to the second part is smaller than the answer to the first part.

Question 8: For $i \in \{1, 2\}$, consider the game G_i, in which two players P_1 and P_2 take turns flipping, independently, a fair coin, where P_i starts. The game ends as soon as heads comes up. The player who flips heads first is the winner of the game G_i. For $j \in \{1, 2\}$, define the event

$$B_{ij} = \text{“}P_j \text{ wins the game } G_i\text{”}.$$

In class (see Section 5.14.2), we have seen that

$$\Pr(B_{11}) = \Pr(B_{22}) = 2/3 \tag{1}$$

and

$$\Pr(B_{12}) = \Pr(B_{21}) = 1/3. \tag{2}$$

Consider the game G, in which P_1 and P_2 take turns flipping, independently, a fair coin, where P_1 starts. The game ends as soon as a second heads comes up. The player who flips the second heads wins the game. Define the event

$$A = \text{"}P_1 \text{ wins the game } G\text{"}.$$

In class (see Section 5.14.3), we used an infinite series to show that

$$\Pr(A) = 4/9. \tag{3}$$

Use the Law of Total Probability to give an alternative proof of (3). You are allowed to use (1) and (2).

Question 9: You would like to generate a uniformly random bit, i.e., with probability 1/2, this bit is 0, and with probability 1/2, it is 1. You find a coin in your pocket, but you are not sure if it is a fair coin: It comes up heads (H) with probability p and tails (T) with probability $1 - p$, for some real number p that is unknown to you. In particular, you do not know if $p = 1/2$. In this question, you will show that this coin can nevertheless be used to generate a uniformly random bit.

Consider the following recursive algorithm GetRandomBit, which does not take any input:

```
Algorithm GetRandomBit:

    // all coin flips made are mutually independent
    flip the coin twice;
    if the result is HT
    then return 0
    else if the result is TH
         then return 1
         else GetRandomBit
         endif
    endif
```

- The sample space S is the set of all sequences of coin flips that can occur when running algorithm GetRandomBit. Determine this sample space S.
- Prove that algorithm GetRandomBit returns a uniformly random bit.