School of Mathematics and Statistics
Carleton University
Math. 2004A, Fall 2013

MOCK TEST 6

Any non-programmable calculator permitted, 1 blank sheet permitted for roughs

Print Name :

Student Number:

Tutorial Section (A1, A4, . . .):

PART I: Multiple Choice Questions
(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[3 marks]** Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F}(x,y,z) = x^2\mathbf{i} + 0\mathbf{j} + 3y^2\mathbf{k}$, $\mathbf{n}$ is an upward unit normal and $\mathcal{S}$ is that part of the plane $x+y+z=1$ in the first octant.

   (a) $1/3$, (b) $1/2$, (c) 0, (d) -2.

   Answer: (a).

   Details: The surface $z = g(x,y) = 1-x-y$ so that $dS = \sqrt{1+g_x^2+g_y^2}\,dA_{xy} = \sqrt{3}\,dA_{xy}$. Next since the surface is a plane, its upward unit normal is given by $\mathbf{n} = (1,1,1)/\sqrt{3}$. On the other hand, the value of $\mathbf{F}$ on the surface is given by $\mathbf{F}(x,y,z) = \mathbf{F}(x,y,1-x-y) = x^2\mathbf{i}+0\mathbf{j}+3y^2\mathbf{k}$, i.e., $\mathbf{F}$ does not change its values on $\mathcal{S}$. Thus, $\mathbf{F}\cdot\mathbf{n}\,dS = (x^2+3y^2)\,dA_{xy}$ Since the projection $\mathcal{R}_{xy}$ of $\mathcal{S}$ onto the $xy-$-plane is a triangle, we get that

   $$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^{1-y}(x^2+3y^2)\,dxdy = \int_0^1\int_0^{1-x}(x^2+3y^2)\,dydx = 1/3.$$

2. **[3 marks]** Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F}(x,y,z) = 3x^2\mathbf{i} + y^2\mathbf{j} + 0\mathbf{k}$ and $\mathcal{S}$ is that part of the plane $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (2u+3v)\mathbf{k}$ with $0 \le u \le 2$ and $-1 \le v \le 1$.

   (a) -32, (b) 24, (c) -36, (d) 12.

   Answer: (c).

   Details: The surface is given parametrically by $\mathbf{r}(u,v)$. The normal vector is given by $\mathbf{r}_u \times \mathbf{r}_v = (-2,3,1)$ where $\mathbf{r}_u = (1,0,2)$ and $\mathbf{r}_v = (0,1,3)$. The corresponding unit normal is then $\mathbf{n} = (-2,3,1)/\sqrt{14}$. But, on $\mathcal{S}$, $\mathbf{F}(\mathbf{r}(u,v)) = (3u^2, v^2, 0)$. In addition, $dS = |\mathbf{r}_u \times \mathbf{r}_v|\,dA_{uv} = \sqrt{14}\,dA_{uv}$. Thus, $\mathbf{F}\cdot\mathbf{n}\,dS = (-6u^2-3v^2)\,dA_{uv}$ Since the projection $\mathcal{R}_{uv}$ of $\mathcal{S}$ onto the $xy-$-plane is a rectangle, we get that

   $$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{-1}^1\int_0^2(-6u^2-3v^2)\,dudv = -36.$$

3. **[3 marks]** Find the area of the ellipse $x = a\cos\theta, y = b\sin\theta$, $0 \le \theta \le 2\pi$, using the line integral $\frac{1}{2}\int_{\mathcal{C}}(x\,dy - y\,dx)$.

   (a) $\pi a^2 b^2$, (b) $\pi a^2 b$, (c) $\pi a b^2$, (d) πab.

   Answer: (d).

   Details: The parametrization of the ellipse is given at the outset. Thus, $x = a\cos\theta, y = b\sin\theta$ implies that $dx = -a\sin\theta\,d\theta$ and $dy = b\cos\theta d\theta$. It follows that on the ellipse $xdy - ydx = ab(\cos^2\theta + \sin^2\theta) = ab$. Hence the area is given by

   $$\frac{1}{2}\int_{\mathcal{C}}(x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} ab\,d\theta = \pi ab.$$

4. **[3 marks]** Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$, and $\mathcal{S}$ is that part of the sphere $\mathcal{T}$ given by $x^2 + y^2 + z^2 = 4$ where $z \geq 0$..

   (a) $8\pi/3$, (b) $16\pi/3$, (c) π, (d) $2\pi/3$.

   Answer: (b).

   Details: Use the Divergence Theorem. Now div$\mathbf{F} = 1$, so

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{T} \text{div}\mathbf{F}\, dV = \text{Volume}\mathcal{T} = \frac{4}{3}\pi 2^3 \frac{1}{2} = \frac{16\pi}{3}.$$

5. **[3 marks]** Evaluate the surface integral $\iint_{\mathcal{S}} \text{curl}\, \mathbf{F} \cdot \mathbf{n}\, dS$ where $\mathbf{F}(x, y, z) = z^2\mathbf{i} + 5x\mathbf{j} + 0\mathbf{k}$, and $\mathcal{S}$ is the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 1$..

   (a) 4, (b) 2, (c) 0, (d) 5.

   Answer: (d).

   Details: Parametrize the square using $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 1\mathbf{k}$ where $0 \leq u \leq 1$ and $0 \leq v \leq 1$. Then $\mathbf{r}_u \times \mathbf{r}_v = (0, 0, 1)$ where $\mathbf{r}_u = (1, 0, 0)$ and $\mathbf{r}_v = (0, 1, 0)$. The corresponding unit normal is then $\mathbf{n} = (0, 0, 1)$. On $\mathcal{S}$, $\mathbf{F}(\mathbf{r}(u, v)) = (1, 5u, 0)$. In addition, $dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dA_{uv} = dA_{uv}$. In addition, an easy calculation shows that curl $\mathbf{F} = 5\mathbf{k}$. Hence, curl $\mathbf{F} \cdot \mathbf{n}\, dS = 5\, dA_{uv}$ Since the projection $\mathcal{R}_{uv}$ of $\mathcal{S}$ onto the $xy-$-plane is a square, we get that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 5\, du dv = 5.$$

PART II: Show all work here and give details.
No additional pages will be accepted

6. **[7+8 marks]** a) Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr}$ where $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ and $\mathcal{C}$ is the straight line from $(0, 0)$ to (1,1).

Solution: 1.

Details: Observe that if $P = y, Q = x$ then $Q_x = P_y$ so that $\mathbf{F}$ is conservative. It follows that the integral is independent of the path chosen to go from $(0, 0)$ to $(1, 1)$. Let $\mathcal{C}_1$ denote the path from $(0, 0)$ to $(0, 1)$ and $\mathcal{C}_2$ denote the path from $(0, 1)$ to $(1, 1)$. Now $\mathcal{C}_1$ is easily parametrized using $x(t) = 0, y(t) = t$, $0 \le t \le 1$. Also $\mathcal{C}_2$ is easily parametrized using $x(s) = s, y(s) = 1$, $0 \le s \le 1$. Therefore,

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} &= \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_{\mathcal{C}_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt + \int_{\mathcal{C}_2} \mathbf{F}(\mathbf{r}(s)) \cdot \mathbf{r}'(s)\ ds \\
&= \int_0^1 \mathbf{F}(x(t), y(t)) \cdot (x'(t), y'(t))\ dt + \int_0^1 \mathbf{F}(x(s), y(s)) \cdot (x'(s), y'(s))\ ds \\
&= \int_0^1 \mathbf{F}(0, t) \cdot (0, 1)\ dt + \int_0^1 \mathbf{F}(s, 1) \cdot (1, 0)\ ds \\
&= \int_0^1 (t, 0) \cdot (0, 1)\ dt + \int_0^1 (1, s) \cdot (1, 0)\ ds \\
&= 0 + 1 = 1.
\end{aligned}$$

OR

Let $y = x$, $0 \le x \le 1$ be the linear representation of the curve $\mathcal{C}$. Then

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} &= \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^1 \mathbf{F}(x, y) \cdot (dx, dy) \\
&= \int_0^1 \mathbf{F}(x, x) \cdot (dx, dx) \\
&= \int_0^1 2x\, dx = 1.
\end{aligned}$$

b) Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr}$ where $\mathbf{F}(x, y) = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}$ and $\mathcal{C}$ is the circle $x^2 + y^2 = 4$, $z = -3$ using Stokes' Theorem.

Solution: -112π.

Details: In using Stokes' Theorem we let $\mathcal{S}$ be the surface $z = -3 = g(x, y)$ whose boundary is the circle $x^2 + y^2 = 4$. Then the outer pointing normal vector here is $(0, 0, 1) = \mathbf{k}$. Also, $dS = \sqrt{1 + 0^2 + 0^2}\, dA_{xy}$. In addition, it is easy to see that $\operatorname{curl} \mathbf{F} = (-3z(y^2 + xz), 0, z^3 - 1)$. Hence $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = z^3 - 1$. Evaluating this quantity on the surface $z = -3$ we get -28. Finally,

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\mathcal{R}_{xy}} (-28) dA_{xy}.$$

Since $\mathcal{R}_{xy}$ is a circle of radius 2 on the xy-plane we can change to polar coordinates and get

$$\iint_{\mathcal{R}_{xy}} (-28) dA_{xy} = \int_0^{2\pi} \int_0^2 (-28)\, r\, dr\, d\theta = -112\pi.$$

The latter number is also -28 times the area of said circle on the xy-plane.