Solutions

School of Mathematics and Statistics
Carleton University
Math. 2004A, Fall 2014
TEST 5

Only the STUDIO 56 calculator permitted, 1 blank sheet permitted for roughs

Print Name : _____

Student Number: _____

Tutorial Section (A1, A4, ...): _____

PART I: Multiple Choice Questions
(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[3 marks]** Evaluate the line integral $\int_C \nabla \mathbf{f} \cdot d\mathbf{r}$ where $\mathbf{f}(x,y,z) = 3x+2y+z$ and C is the line segment joining the points $P(0,1,1)$ to $Q(1,0,1)$.
$I = f(1,0,1) - f(0,1,1) = 1.$
(a) 1, (b) 0, (c) -1, (d) 2.

2. **[3 marks]** Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F}(x,y,z) = 2x\mathbf{i}+3y\mathbf{j}+4z\mathbf{k}$ and S is the surface of the sphere $x^2+y^2+z^2=4$, $\mathbf{n}$ is an outward unit normal vector to S.
$I = \iiint_T \operatorname{div} F\, dV = 9\iiint_T dV$
$= 9\cdot\frac{4}{3}\pi\cdot 2^3 = 96\pi$
(a) -112π, (b) 4π, (c) 36π, (d) 96π. (e) None of these.

3. **[3 marks]** A vector field $\mathbf{F}(x,y,z)$ has the property that its curl is given by curl $\mathbf{F} = 6z\mathbf{k}$. Let S denote the surface of intersection of the plane $z=2$ with the cylinder $x^2+y^2=9$ having for its boundary the curve C oriented positively (or in the counterclockwise direction). Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ using any method.
Stokes $\Rightarrow I = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS$
$= \iint_S 6z\vec{k}\cdot\vec{k}\,dS \xrightarrow{z=2} 12\iint_S dS = 12\cdot(\pi 3^2) = 108\pi$
(a) 98π, (b) 108π, (c) 36π, (d) 216π, (e) None of these.

4. **[3 marks]** Evaluate the line integral $\int_C P(x,y)\,dx + Q(x,y)\,dy$ where $P(x,y) = 6y+e^y$ and $Q(x,y) = xe^y$ where C is the ellipse given by $2x^2+3y^2=36$ oriented in a counterclockwise direction.
Use Green's Thm. $Q_x - P_y = -6$.
$\therefore \int P\,dx+Q\,dy = -6\iint_R dA_{xy}$
$= -6\cdot$ Area of ellipse
$= -6\left(\pi\sqrt{18}\sqrt{12}\right) = -36\pi\sqrt{6}.$
(a) $2\pi\sqrt{6}$, (b) $-36\pi\sqrt{6}$, (c) 0, (d) -6π, (e) None of these.

5. **[3 marks]** The vector field given by $\mathbf{F} = (y-x)\mathbf{i}+(x+y)\mathbf{j}+\mathbf{k}$ is conservative.
(curl $\vec{F} = \vec{0}$)
(a) True, (b) False.

PART II: Show all work here and give details.
No additional pages will be accepted

6. **[7+8 marks]** a) Show that the vector field $\mathbf{F}(x,y,z) = 2xy^3z^4\mathbf{i}+3x^2y^2z^4\mathbf{j}+4x^2y^3z^3\mathbf{k}$ is conservative. Next, find a scalar function $f(x,y,z)$ with $\nabla f = \mathbf{F}$.

a) Show that curl $\vec{F} = \vec{0}$ ← (2).
b) Write $\nabla f = \vec{F} \Rightarrow \frac{\partial f}{\partial x} = 2xy^3z^4,\ \frac{\partial f}{\partial y} = 3x^2y^2z^4,\ \frac{\partial f}{\partial z} = 4x^2y^3z^3$ (3)
$f_x = 2xy^3z^4 \Rightarrow f = x^2y^3z^4 + g(y,z)$ (1)

Next, $f_y = 3x^2y^2z^4 + g_y(y,z) = 3x^2y^2z^4 \Rightarrow g_y(y,z) = 0$.

$\Rightarrow g(y,z) = h(z)$ only $\therefore \boxed{f = x^2y^3z^4 + h(z)}$ ← (1)

Finally, $f_z = 4x^2y^3z^3 + h'(z) = 4x^2y^3z^3 \Rightarrow h'(z) = 0$

$\Rightarrow \boxed{h(z) = \text{constant}}$ (1)

$\therefore \boxed{f = x^2y^3z^4 + C}$ is required fn. (2)

b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = (z-y)\mathbf{i} - (x+z)\mathbf{j} - (x+y)\mathbf{k}$ and C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = 4$ with the plane $z = y$ oriented counterclockwise (as viewed from above) using Stokes' Theorem.

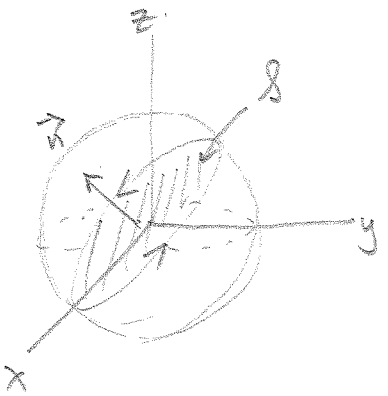
→ Stokes ⇒ $\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\vec{n}\, dS}$

Here, $\boxed{\text{curl}\vec{F} = 2\vec{j}}$ (1) → Next, the curve of intersection is given by ($\because z = y$)

$x^2+y^2+y^2 = 4 \Rightarrow x^2+2y^2 = 4 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$, an ellipse!

Thus S is the interior of this ellipse, $S = \{(x,y,z) : x^2+2y^2 \le 4, z = y\}$ & its projection on xy-plane, $R_{xy} = \{(x,y) : x^2+2y^2 \le 4\}$.

Now the normal $\vec{n}$ to $z = y$ is given by either $\pm(-g_x, -g_y, 1)$ since $g(x,y) = y$ here, i.e., either $\pm(0,-1,1)$. By the right hand rule we see that $\boxed{(0,-1,+1)}$ is the correct choice. ← (1)



(1) → $\boxed{\vec{n} = \frac{(0,-1,1)}{\sqrt{2}}}$ is the unit vector we want.

$\because dS = \sqrt{1+g_x^2+g_y^2}\, dA_{xy} \Rightarrow \boxed{\sqrt{2}\, dA_{xy} = dS}$ ← (1)

$$\iint_S \text{curl}\vec{F}\cdot\vec{n}\, dS = \iint_{R_{xy}} (0,2,0)\cdot\frac{(0,-1,1)}{\sqrt{2}}\sqrt{2}\, dA_{xy}$$

$$= -2\iint_{R_{xy}} dA_{xy} = -2(\text{area of } R_{xy})$$

$$= -2(\text{area of ellipse}) = -2(\pi\sqrt{2}\cdot 2)$$

$$= \boxed{-4\pi\sqrt{2}}$$ (2)