Solutions

School of Mathematics and Statistics
Carleton University
Math. 2004A, Fall 2014

TEST 4

Only the STUDIO 56 calculator permitted, 1 or more blank sheets permitted for roughs (not to be attached here)

Print Name :

Student Number:

Tutorial Section (A1, A4, ...):

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. [**3 marks**] Find the Jacobian of the transformation $x = 2u + v$, $y = u^2 - v$. (a) 2, (b) $v - u^2$, (c) $-2(u + 1)$, (d) $v + 1$, (e) none of these.
   Find determinant of Jacobian matrix

2. [**3 marks**] Let $\mathcal{R}$ be the region bounded by the graphs of $x = y^2$, $x = 0$ and $y = 1$. The change of variable $x = u^2$, $y = v$ transforms the integral $\iint_{\mathcal{R}} y \sin x \, dA$ into which one of the following integrals?
   (a) $\int_0^1 \int_1^{2v} uv \sin u^2 \, dudv$, (b) $\int_0^1 \int_0^1 v \sin u^2 \, dudv$, (c) $\int_0^1 \int_0^2 \sin u^2 \, du\, dv$, (d) $\int_0^1 \int_0^v 2uv \sin u^2 \, du\, dv$, (e) none of these.

3. [**3 marks**] What is the divergence of the vector field $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ at the point $(1, 0, -1)$?
   (a) 0, (b) 1, (c) -1, (d) 2, (e) none of these. (div = 0 always here).

4. [**3 marks**] Let $\mathcal{R}$ denote the sphere in $\mathbf{R}^3$ centered at the origin and radius equal to 1 (a.k.a. the unit sphere). Which of the following integrals represents the triple integral

$$\iiint_{\mathcal{R}} e^{-\sqrt{x^2+y^2+z^2}} \, dV$$

   in spherical coordinates?
   (a) $\int_0^{\pi} \int_0^{\pi} \int_0^1 e^{-\rho} \sin \phi d\rho \, d\phi \, d\theta$, (b) $\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 e^{-\rho} \sin \theta d\rho \, d\phi \, d\theta$, (c) $\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 e^{-\rho} \sin \phi d\rho \, d\phi \, d\theta$,
   (d) $\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho e^{-\rho^2} \sin \phi d\rho \, d\phi \, d\theta$, (e) none of these

5. [**3 marks**] Does there exist a vector field $\mathbf{F}$ whose curl is given by the quantity curl $\mathbf{F} = xy\mathbf{i} - yz\mathbf{j} + xy\mathbf{k}$.?
   (a) Yes, (b) No, (b/c div(curl F) ≠ 0 yet div(curl F) = 0 if such a field did exist)

PART II: Show all work here and give details.
No additional pages will be accepted

6. [**7 marks**] a) Evaluate the line integral $\int_C (x^2 - y) \, dx + (y^2 + x) \, dy$ along the straight line from $(0, 1)$ to $(1, 2)$.

C is given by setting $\boxed{y = x+1}$ (this defines C) where $\boxed{0 \le x \le 1}$

Method 1: $I = \int_0^1 \underbrace{(x^2 - (x+1))\,dx}_{1} + (\underbrace{(x+1)^2 + x}_{1})\,dx$ $\quad$ ($\because$ $y = x+1$ on C, $dy = dx$)

$= \underbrace{\int_0^1 (2x^2 + 2x)\,dx}_{1} = 5/3$ 1

Method 2: The line joining $P(0,1)$ to $Q(1,2)$ is given parametrically by C: $(1-t)P + tQ = (1-t)(0,1) + t(1,2) = (0, 1-t) + (t, 2t)$

$= \boxed{(t, 1+t) = (x(t), y(t))}$ 2, $\boxed{0 \le t \le 1}$ 1

$\therefore I = \int_0^1 (t^2 - (t+1)) \cdot 1\,dt + ((1+t)^2 + t) \cdot 1\,dt$

$= \cdots = 5/3$ as before.

7. **[8 marks]** Evaluate the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k}$ along the path C given parametrically by $x = t, y = t^2, z = t^3$, where $0 \le t \le 1$.

Here $\underbrace{\vec{r}(t) = (t, t^2, t^3)}_{(1)}, \quad \underbrace{0 \le t \le 1}_{(1)}.$

$\therefore \underbrace{\vec{r}'(t) = (1, 2t, 3t^2)}_{(1)} = (x(t), y(t), z(t)).$

$$I = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \leftarrow (1)$$

$$= \int_0^1 \vec{F}(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))\, dt$$

$$= \int_0^1 \underbrace{(3t^2 - 6t^5)}_{(1)} dt + \underbrace{(4t^3 + 6t^5)}_{(1)} dt + \underbrace{(3t^2 - 12t^{11})}_{(1)} dt$$

$$= 2.$$

(1)