Solutions

School of Mathematics and Statistics
Carleton University
Math. 2004A, Fall 2014

TEST 2

Only the STUDIO 56 calculator permitted, 1 or more blank sheets permitted for roughs (not to be attached here)

Print Name :

Student Number:

Tutorial Section (A1, A4, ...):

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. [**2 marks**] Let $f(x,y)=\sqrt{x^2-y^3}$. Find $\frac{\partial f}{\partial x}$ at $(x,y)=(2,0)$.

   (a) -1, (b) 1, (c) 0, (d) 2, (e) None of these.

   $f_x = \frac{x}{\sqrt{x^2-y^3}}$, Set $x=2, y=0$

2. [**2 marks**] Let $f(x,y)=x\,e^{y/x}$. Find $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ at $(x,y)=(1,-1)$.

   (a) e, (b) -1, (c) 1, (d) $1/e$.

   $f_y = e^{y/x}$ $\quad f_{xy} = -y/x^2\, e^{y/x}$
   Set $x=1, y=-1$.

3. [**2 marks**] Find the equation of the tangent plane to the point $P(0,3,0)$ on the surface defined by $x^2+4y^2+9z^2=36$.

   (a) $y=3$, (b) $x+2y+z=6$, (c) $z=2$, (d) $x=6$.

   $f(x,y,z) = x^2+4y^2+9z^2-36=0$
   $\nabla f(0,3,0) = (0,24,0) \therefore 24y = 72 \Rightarrow y=3.$

4. [**2 marks**] Find the value of the directional derivative of the function f defined by $f(x,y,z)=x^2+y^2+z^2$ at $P(1,0,1)$ in the direction of the vector $\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

   (a) $\sqrt{6}$, (b) 4, (c) 0, (d) $2/\sqrt{6}$.

   $\vec{u} = \frac{i-j+2k}{\sqrt{6}}$ is a unit vector in dir. of $(1,-1,2)$
   $\nabla f = (2x,2y,2z)\big|_{(1,0,1)} = (2,0,2) \therefore \nabla f\cdot\vec{u} = (2,0,2)\cdot(1,-1,2)/\sqrt{6} = \sqrt{6}$

5. [**2 marks**] The limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}=1$.

   (a) TRUE, (b) FALSE,

   F Try $y=0$ and $y=2x$. Get different limits!

PART II: Show all work here and give details. No additional pages will be accepted

6. [**5+5 marks**] a) Find the equation of the normal line to the surface given by $z=9x^2+4y^2$ at $P(-1,2,25)$.

   b) Let $u(x,y)=\frac{x}{x^2+y^2}$ where $x=\sec 2t$ and $y=\tan t$. Use the Chain Rule to evaluate $\frac{du}{dt}$ at $t=0$.

a) Normal line to a surface $f(x,y,z)=0$ at $P(x_0,y_0,z_0)$ is given by (1/2)

(1) $\boxed{\frac{x-x_0}{\partial f/\partial x(P)} = \frac{y-y_0}{\partial f/\partial y(P)} = \frac{z-z_0}{\partial f/\partial z(P)}}$. But $\boxed{\frac{\partial f}{\partial x}=-18x}$, $\boxed{\frac{\partial f}{\partial y}=-8y}$, (1/2)

(1/2) $\boxed{\frac{\partial f}{\partial z}=1}$ (here $f(x,y,z) = z-9x^2-4y^2$). $\therefore \nabla f = (-18x,-8y,1)$

$\Rightarrow$ at P, $\nabla f(P) = ((-18)(-1),(-8)(2),1) = \boxed{(18,-16,1)}$ (1)

$(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})$ $\therefore \boxed{\frac{x+1}{18} = \frac{y-2}{-16} = \frac{z-25}{1}}$, or $\begin{cases} x=-1+18t \\ y=2-16t \\ z=25+t \end{cases}$.

or $\frac{x+1}{-18} = \frac{y-2}{16} = \frac{z-25}{-1}$.

b) $\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}}$ (2)

$t=0 \Rightarrow x=1, y=0$

$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$ (1/2) $\quad$ (1/2) $\frac{dx}{dt} = 2\sec 2t\tan 2t$

$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$ (1/2) $\quad$ (1/2) $\frac{dy}{dt} = \sec^2 t$

$\Downarrow$

Answer: $\underline{\underline{0}}$ (1/2)

7. [**5 +5 marks**] a) Does the following limit exist? Explain. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2}$.

b) Let $f(x,y) = \sqrt{2x+3y^2}$. Find a vector **u** giving the direction in which f increases most rapidly at the point $P(3,2)$. What is its maximum rate of increase? (Hint: Use the gradient vector.)

a) No. ← 1

Coming into $(0,0)$ using curve $\boxed{(x,x)}$ (or $y=x$) we get the limit, (as $x\to 0$) $\frac{3x^2}{3x^2+x^2} \to \frac{3}{4}$. 1 1

or using curve $\boxed{(0,y)}$ we get limit "0". 1

Since these limits are different, the limit cannot exist! 2

N.B. Of course any other 2 curves giving different limits at $(0,0)$ could be used.

---

b). $\nabla f(P) = \left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)\right) = \left(\frac{1}{\sqrt{2x+3y^2}}, \frac{3y}{\sqrt{2x+3y^2}}\right)$ 1 1

At $P(3,2)$, $\boxed{\nabla f(P) = \left(\frac{1}{\sqrt{18}}, \frac{6}{\sqrt{18}}\right) = \vec{u}.}$ is direction of fastest increase. ← 1

max. rate $= |\nabla f(P)| = \sqrt{\frac{1}{18}+\frac{36}{18}} = \sqrt{\frac{37}{18}}$ 1 1