University of Ottawa
MAT 1332 Midterm solutions

Question 1. [5 points] Consider the matrix:

(a) Find the eigenvalues of A.

(b) Give an eigenvector for each of the eigenvalues found in (a).

Version A.

$$A = \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}$$

(a) The characteristic polynomial is $p_A(\lambda) = \det(\lambda I - A) = (\lambda + 5)(\lambda - 4) - (-3)6 = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2)$. The eigenvalues are the roots of the characteristic polynomial, i.e. $\lambda_1 = -2$ and $\lambda_2 = 1$ (**1 Point**).

(b) To find the eigenvectors for λ_1, solve $(-2I - A)\vec{x} = \vec{0}$, ie in augmented matrix form,

$$\left(\begin{array}{cc|c} 3 & 6 & 0 \\ -3 & -6 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus $x = -2y$ (**1 Point**), and we can take $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (**1 Point**). To find the eigenvectors for $\lambda 2$, solve $(I - A)\vec{x} = \vec{0}$, ie in augmented matrix form,

$$\left(\begin{array}{cc|c} 6 & 6 & 0 \\ -3 & -3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus, $x = -y$ (**1 Point**), and we can take $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (**1 Point**).

Version B.

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$$

(a) The characteristic polynomial is $p_A(\lambda) = \det(\lambda I - A) = (\lambda - 5)(\lambda + 4) + 18 = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$. The eigenvalues are the roots of the characteristic polynomial, i.e. $\lambda_1 = -1$ and $\lambda_2 = 2$ (**1 Point**).

(b) To find the eigenvectors for λ_1, solve $(-I - A)\vec{x} = \vec{0}$, ie in augmented matrix form,

$$\left(\begin{array}{cc|c} -6 & -3 & 0 \\ 6 & 3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus $x = -1/2y$ (**1 Point**), and you can take $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (**1 Point**). To find the eigenvectors for λ_2, solve $(2I - A)\vec{x} = \vec{0}$, ie in augmented matrix form,

$$\left(\begin{array}{cc|c} -3 & -3 & 0 \\ 6 & 6 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus $x = -y$ (**1 Point**), and you can take $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (**1 Point**).

Version C.

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$$

(a) The characteristic polynomial is $p_A(\lambda) = \det(\lambda I - A) = (\lambda - 3)\lambda + 2 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$. The eigenvalues are the roots of the characteristic polynomial,$\lambda_1 = 1$ and $\lambda_2 = 2$ (**1 Point**).

(b) To find the eigenvectors for λ_1, solve $(I - A)\vec{x} = \vec{0}$, ie in augmented matrix form,

$$\left(\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus $x = -1/2y$ (**1 Point**), and you can take $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (**1 Point**). To find the eigenvectors for λ_2, solve $(2I - A)\vec{x} = \vec{0}$, ie in augmented matrix form

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus $x = -y$ (**1 Point**), and you can take $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (**1 Point**).

Version D.

(a) The characteristic polynomial is $p_A(\lambda) = \det(\lambda I - A) = (\lambda - 4)(\lambda + 5) - (-3)6 = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1)$.The eigenvalues are the roots of the characteristic polynomial, i.e. $\lambda_1 = -2$ and $\lambda_2 = 1$. (**1 Point**)

(b) To find the eigenvectors for λ_1, solve $(-2I - A)\vec{x} = \vec{0}$, ie in augmented matrix form

$$\left(\begin{array}{cc|c} -6 & 6 & 0 \\ -3 & 3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus $x = y$ (**1 Point**), and you can take $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (**1 Point**). To find the eigenvectors for λ_2, solve $(I - A)\vec{x} = \vec{0}$, ie in augmented matrix form

$$\left(\begin{array}{cc|c} -3 & 6 & 0 \\ -3 & 6 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus $x = 2y$ (**1 Point**), and you can take $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (**1 Point**).

Question 2. [6 points] Consider the following differential equation

(a) Find the equilibrium points of the differential equation. y' = f(y)

(b) Classify the equilibrium points in (a) (stable or unstable).

(c) Draw the phase-line diagram of the differential equation.

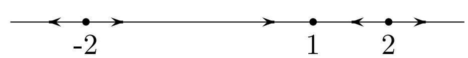
(d) Sketch in the same coordinate system the equilibrium solutions and the solution with initial condition $y(0) = 0$.

Version A. $a = 1$, $b = 2$ and $C = 0$.

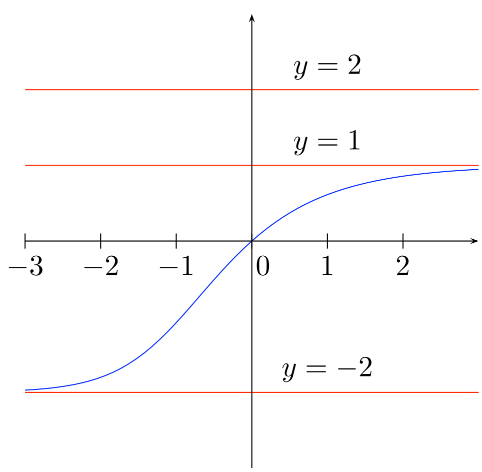
(a) We have $y' = f(y)$ with $f(y) = (y-1)(y^2-4) = (y-1)(y-2)(y+2)$. Hence the equilibria are $-2, 1, 2$ (**1 Point**).

(b) To decide about stability calculate the derivative $f'(y) = 3y^2 - 2y - 4$ (**1 Point**) and evaluate at each equilibrium $f'(-2) = 12 > 0$, $f'(1) = -3 < 0$ and $f'(2) = 4 > 0$. Hence, -2 and 2 are unstable and 1 is stable (**1 Point**).

(c) (**1 Point**)



(d) (**2 Points**)

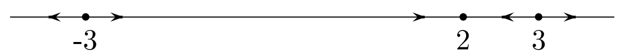


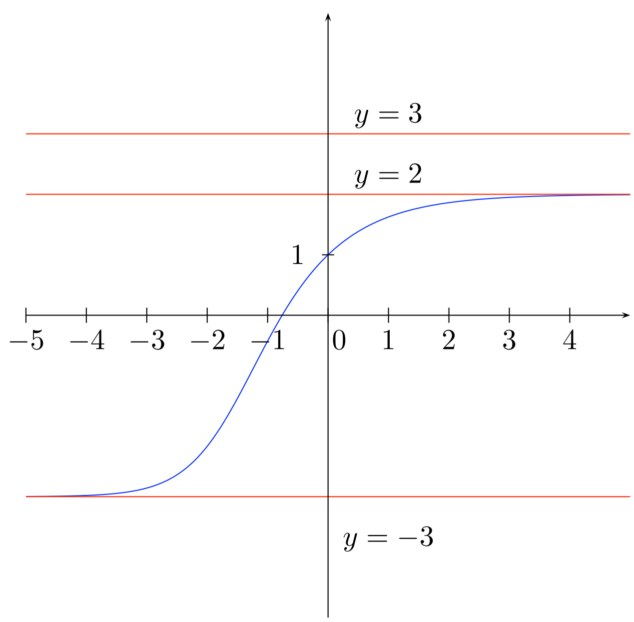
Version B. $a = 2$, $b = 3$, $C = 1$.

(a) We have $y' = f(y)$ with $f(y) = (y-2)(y^2-9) = (y-2)(y-3)(y+3)$. Hence the equilibria are $-3, 2, 3$ (**1 Point**).

(b) To decide about stability calculate the derivative $f'(y) = 3y^2 - 4y - 9$ (**1 Point**) and evaluate at each equilibrium $f'(-3) = 30 > 0$, $f'(2) = -5 < 0$ and $f'(3) = 6 > 0$. Hence, -3 and 3 are unstable and 2 is stable (**1 Point**).

(c) (**1 Point**) (d) (**2 Points**)



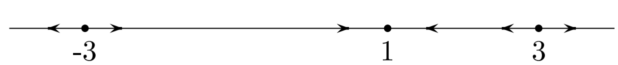


Version C. $a = 1$, $b = 3$ and $C = 2$.

(a) We have $y' = f(y)$ with $f(y) = (y-1)(y^2-9) = (y-1)(y-3)(y+3)$. Hence the equilibria are $-3, 1, 3$ (**1 Point**).

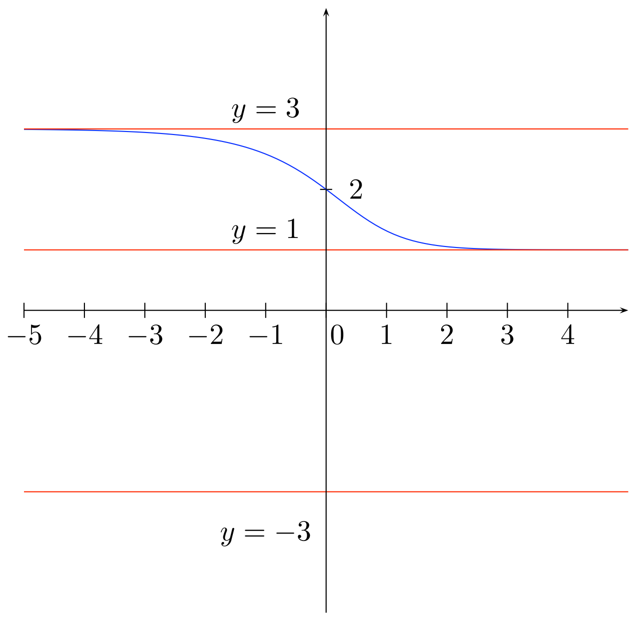
(b) To decide about stability calculate the derivative, $f'(y) = 3y^2 - 2y - 9$ (**1 Point**) and evaluate at each equilibrium $f'(-3) = 24 > 0$, $f'(1) = -8 < 0$ and $f'(3) = 12 > 0$. Hence, -3 and 3 are unstable and 1 is stable (**1 Point**).

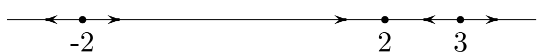
(c) (**1 Point**) (d) (**2 Points**) **Version D.** (a) On a $y' = f(y)$ with $f(y) = (y-3)(y^2-4) =$

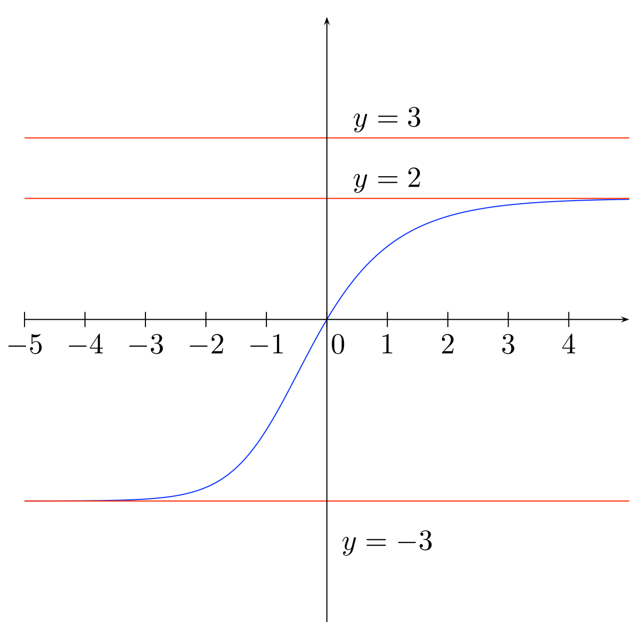


$(y-3)(y-2)(y+2)$. Hence the equilibria are $-2, 2, 3$ (**1 Point**).

(b) To decide about stability calculate the derivative, $f'(y) = 3y^2 - 6y - 4$ (**1 Point**) and evaluate at each equilibrium $f'(-2) = 20 > 0$, $f'(2) = -4 < 0$ and $f'(3) = 5 > 0$. Hence, -2 and 3 are unstable and 2 is stable (**1 Point**).


$y = 3$
2
$y = 1$
-5 -4 -3 -2 -1 0 1 2 3 4
$y = -3$


-2
2
3


$y = 3$
$y = 2$
-5 -4 -3 -2 -1 0 1 2 3 4
$y = -3$

(c) (**1 Point**) (d) (**2 Point**)

Question 3. [5 Points] Consider the following equation

$$\frac{1}{2}z^2 + az + b = 0.$$

(a) Solve this equation in $\mathbb{C}$. Write the solutions in the form $a+b$i and in polar coordinates.

(b) Sketch the solutions in the complex plane.

Version A. $a = -1$, $b = 1$

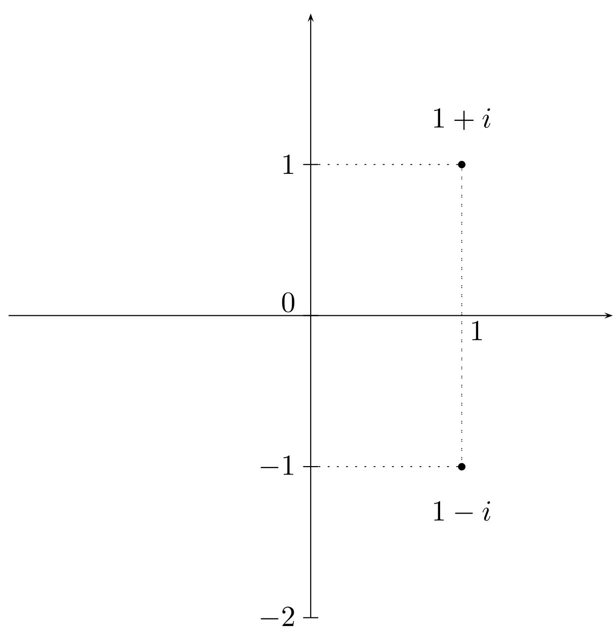
(a) $\Delta = 1 - 2 = -1$ and we have therefore the following equations (**2 Points**)

$$z_1 = \frac{-(-1) + \sqrt{-1}}{2\frac{1}{2}} = 1 + i \quad \text{and} \quad z_2 = \frac{-(-1) + \sqrt{-1}}{2\frac{1}{2}} = 1 - i.$$

For the polar form, calculate $|1 + i| = |1 - i| = \sqrt{2}$ and $\arctan(1) = \pi/4$, $\arctan(-1) = -\pi/4$ and d (**2 Points**)

$$z_1 = \sqrt{2}e^{i\pi/4} \quad \text{and} \quad z_2 = \sqrt{2}e^{-i\pi/4}.$$

(b) (**1 Point**)



Version B. $a = -1$, $b = 2$

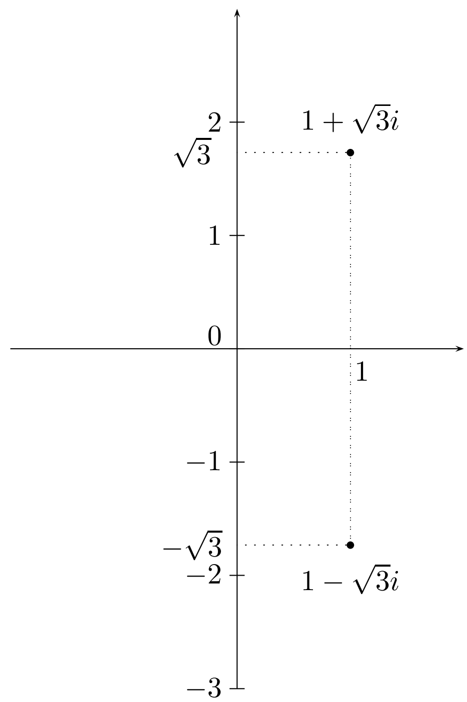
(a) $\Delta = 1 - 4 = -3$ and we have therefore the following equations (**2 Points**)

$$z_1 = \frac{-(-1) + \sqrt{-3}}{2\frac{1}{2}} = 1 + \sqrt{3}i \quad \text{and} \quad z_2 = \frac{-(-1) + \sqrt{-3}}{2\frac{1}{2}} = 1 - \sqrt{3}i.$$

For the polar form, calculate $|1+\sqrt{3}i| = |1-\sqrt{3}i| = \sqrt{4} = 2$ and $\arctan(\sqrt{3}) = \pi/3$, $\arctan(-\sqrt{3}) = -\pi/3$ and hence (**2 Points**)

$$z_1 = 2e^{i\pi/3} \quad \text{and} \quad z_2 = 2e^{-i\pi/3}.$$

(b) (**1 Point**)



Version C. $a = 1$, $b = 1$

(a) $\Delta = 1 - 2 = -1$ and we have therefore the following equations (**2 Points**)

$$z_1 = \frac{-1+\sqrt{-1}}{2\frac{1}{2}} = -1+i \quad \text{and} \quad z_2 = \frac{-1+\sqrt{-1}}{2\frac{1}{2}} = -1-i.$$

For the polar form, calculate $|-1+i| = |-1-i| = \sqrt{2}$ and $\arctan(1) = \pi/4$, $\arctan(-1) = -\pi/4$ and hence (since the x-coordinate is < 0) (**2 Points**)

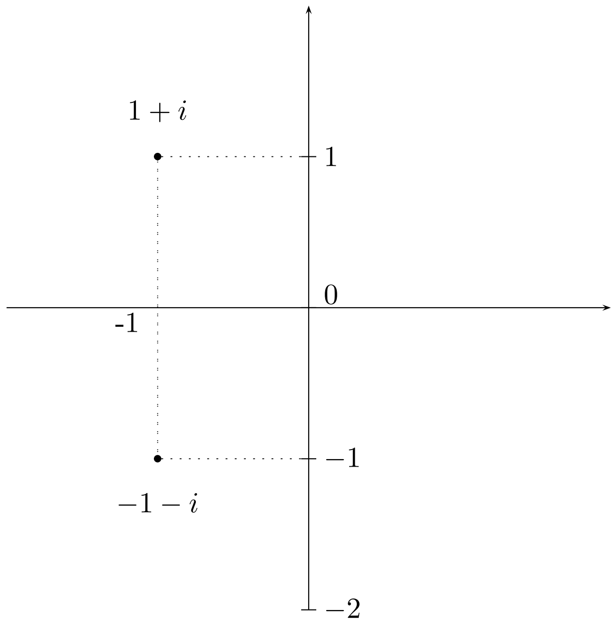
$$z_1 = \sqrt{2}e^{i3\pi/4} \quad \text{and} \quad z_2 = \sqrt{2}e^{-3i\pi/4}.$$

(b) (**1 Point**)

Version D. $a = -1$, $b = 1$

(a) $\Delta = 1 - 4 = -3$ and we have therefore the following equations (**2 Points**)

$$z_1 = \frac{-1+\sqrt{-3}}{2\frac{1}{2}} = -1+\sqrt{3}i \quad \text{et} \quad z_2 = \frac{-1-\sqrt{-3}}{2\frac{1}{2}} = -1-\sqrt{3}i.$$



For the polar form, calculate $|-1+\sqrt{3}i| = |-1-\sqrt{3}i| = \sqrt{4} = 2$ and $\arctan(-\sqrt{3}) = -\pi/3$, $\arctan(\sqrt{3}) = \pi/3$. hence (since the x-coordinate is < 0) (**2 Points**)

$$z_1 = 2e^{i2\pi/3} \quad \text{and} \quad z_2 = e^{-i2\pi/3}.$$

(b) **1 Point**

Question 4. [8 Points] **Version A.**

(a) Find all values of a such that the matrix below is invertible.

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & -2 & -a \\ 3 & a & -2 \end{pmatrix}$$

(b) Find the inverse when $a = 0$.

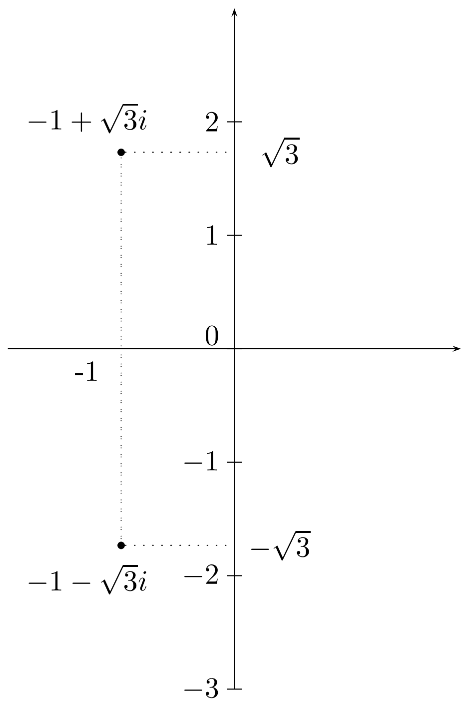
(c) Show that $\lambda = -2$ is an eigenvalue when $a = 0$.

(d) Find all eigenvectors corresponding to $\lambda = -2$ when $a = 0$.

(a) We compute the determinant of A (**1 Point**)

$$\det(A) = 4 + 0 - 2a - 6 + a^2 + 0 = a^2 - 2a - 2.$$

Hence $\det(A) = 0 \iff a = \frac{2\pm\sqrt{12}}{2} = 1 \pm \sqrt{3}$. Since A is invertible iff $\det(A) \neq 0$, we obtain that A is invertible iff $a \neq 1 \pm \sqrt{3}$ (**1 Point**).



(b) (**2 Points**) Consider the augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 & 0 \\ 3 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \overset{R_2-2R_1\to R_2,\ R_3-3R_1\to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\overset{-1/2R_2\to R_2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right) \overset{R_1+R_3\to R_1,\ R_2+R_3\to R_2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -2 & -1/2 & 1 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

and hence

$$A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ -2 & -1/2 & 1 \\ -3 & 0 & 1 \end{pmatrix}$$

(c) Calculate the characteristic polynomial (**1 Point**)

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda-1)(\lambda+2)(\lambda+2) - 3(\lambda+2)$$

and it easy to verify that $p_A(-2) = 0$ (**1 Point**).

(d) Solve $(-2I - A)\vec{x} = \vec{0}$ (**1 Point**)

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, y is free and $x = z = 0$, and the set of eigenvectors is (**1 Point**)

$$\left\{ \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}.$$

Version B.

(a) Find all values of a such that the matrix below is invertible.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & a \\ -1 & -a & 1 \end{pmatrix}$$

(b) Find the inverse when $a = 0$.

(c) Show that $\lambda = -2$ is an eigenvalue when $a = 0$.

(d) Find all eigenvectors corresponding to $\lambda = -2$ when $a = 0$.

(a) We compute the determinant of A (**1 Point**)

$$\det(A) = -2 - 2a + 0 - 2 + a^2 + 0 = a^2 - 2a - 4.$$

Hence $\det(A) = 0 \iff a = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$. Since A is invertible iff $\det(A) \neq 0$, we obtain that A is invertible iff $a \neq 1 \pm \sqrt{5}$ (**1 Point**).

(b) (**2 Points**) Consider the augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \overset{R_3+R_1 \to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array}\right)$$

$$\overset{R_3+R_2 \to R_3,\ R_2/2 \to R_2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right) \overset{1/2 R_3 \to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right)$$

$$\overset{R_1 - R_3 - 2R_2 \to R_1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right)$$

and hence

$$A^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 0 & -1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

(c) Calculate the characteristic polynomial (**1 Point**)

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)(\lambda + 2)(\lambda - 1) + (\lambda + 2)$$

and it easy to verify that $p_A(-2) = 0$ (**1 Point**).

(d) Solve $(-2I - A)\vec{x} = \vec{0}$ (**1 Point**)

$$\left(\begin{array}{ccc|c} -3 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ -3 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & -2 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, z is free and $y = -5z$ and $x = 3z$, and the set of eigenvectors is (**1 Point**)

$$\left\{ \begin{pmatrix} 3t \\ -5t \\ t \end{pmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}.$$

Version C.

(a) Find all values of a such that the matrix below is invertible.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & a \\ -2 & -a & 1 \end{pmatrix}$$

(b) Find the inverse when $a = 0$.

(c) Show that $\lambda = -1$ is an eigenvalue when $a = 0$.

(d) Find all eigenvectors corresponding to $\lambda = -1$ when $a = 0$.

(a) We compute the determinant of A (**1 Point**)

$$\det(A) = -1 + 0 - a - 2 + a^2 + 0 = a^2 - a - 3.$$

Hence $\det(A) = 0 \iff a = \frac{1 \pm \sqrt{13}}{2}$. Since A is invertible iff $\det(A) \neq 0$, we obtain that A is invertible iff $a \neq \frac{1 \pm \sqrt{13}}{2}$ (**1 Point**).

(b) (**2 Points**) Consider the augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \overset{R_2 - R_1 \to R_2,\ R_3 + 2R_1 \to R_1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & 0 & 1 \end{array}\right)$$

$$\overset{-R_2 \to R_2,\ 1/3R_3 \to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2/3 & 0 & 1/3 \end{array}\right) \overset{R_2 - R_3 \to R_2,\ R_1 - R_3 \to R_1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & -1/3 \\ 0 & 1 & 0 & 1/3 & -1 & -1/3 \\ 0 & 0 & 1 & 2/3 & 0 & 1/3 \end{array}\right)$$

and hence

$$A^{-1} = \begin{pmatrix} 1/3 & 0 & -1/3 \\ 1/3 & -1 & -1/3 \\ 2/3 & 0 & 1/3 \end{pmatrix}$$

(c) Calculate the characteristic polynomial (**1 Point**)

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)(\lambda + 1)(\lambda - 1) + 2(\lambda + 1)$$

and it easy to verify that $p_A(-1) = 0$ (**1 Point**).

(d) Solve $(-I - A)\vec{x} = \vec{0}$ (**1 Point**)

$$\left(\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, y is free and $x = z = 0$, and the set of eigenvectors is (**1 Point**)

$$\left\{ \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}.$$

Version D.

Question 4. [8 Points]

(a) Find all values of a such that the matrix below is invertible.

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & a \\ 1 & -a & 2 \end{pmatrix}$$

(b) Find the inverse when $a = 0$.

(c) Show that $\lambda = -1$ is an eigenvalue when $a = 0$.

(d) Find all eigenvectors corresponding to $\lambda = -1$ when $a = 0$.

(a) We compute the determinant of A (**1 Point**)

$$\det(A) = -2 - a + 0 + 1 + a^2 + 0 = a^2 - a - 1.$$

Hence $\det(A) = 0 \iff a = \frac{1 \pm \sqrt{5}}{2}$. Since A is invertible iff $\det(A) \neq 0$, we obtain that A is invertible iff $a \neq \frac{1 \pm \sqrt{5}}{2}$ (**1 Point**).

(b) (**2 Points**) Consider the augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \overset{R_3 - R_1 \to R_1,\ -R_2 \to R_2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{R_3-R_2\to R_3}{\rightsquigarrow}\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & -1 & 1 & 1\end{array}\right)\overset{R_1+R_2-R_3\to R_1}{\rightsquigarrow}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 2 & -2 & -1\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & -1 & 1 & 1\end{array}\right)$$

and so

$$A^{-1}=\begin{pmatrix}2 & -2 & -1\\ 0 & -1 & 0\\ -1 & 1 & 1\end{pmatrix}$$

(c) Calculate the characteristic polynomial (**1 Point**)

$$p_A(\lambda)=\det(\lambda I-A)=(\lambda-1)(\lambda+1)(\lambda-2)-(\lambda+1).$$

and it is easy to verify that $p_A(-1)=0$ (**1 Point**).

(d) Solve$(-I-A)\vec{x}=\vec{0}$ (**1 Point**)

$$\left(\begin{array}{ccc|c}-2 & 1 & -1 & 0\\ -1 & 0 & 0 & 0\\ 2 & 0 & 0 & 0\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

Thus, z is free and $y=z$ and $x=0$, and the set of eigenvectors is (**1 Point**)

$$\left\{\begin{pmatrix}0\\ t\\ t\end{pmatrix}\mid t\in\mathbb{R}, t\neq 0\right\}.$$

Question 5. [5 Points]
Version A.

Solve the linear system

$$\begin{array}{ccccccc}3x & + & 2y & + & 4z & = & 1\\ 2x & + & y & + & 2z & = & -1\\ x & + & 2y & + & 4z & = & 7\end{array}$$

Solution. We write the augmented matrix and use row reduction (**3 Points**)

$$\left(\begin{array}{ccc|c}3 & 2 & 4 & 1\\ 2 & 1 & 2 & -1\\ 1 & 2 & 4 & 7\end{array}\right)\overset{R_1\leftrightarrow R_3}{\rightsquigarrow}\left(\begin{array}{ccc|c}1 & 2 & 4 & 7\\ 2 & 1 & 2 & -1\\ 3 & 2 & 4 & 1\end{array}\right)\overset{R_2-2R_1\to R_2,\ R_3-3R_1\to R_3}{\rightsquigarrow}\left(\begin{array}{ccc|c}1 & 2 & 4 & 7\\ 0 & -3 & -6 & -15\\ 0 & -4 & -8 & -20\end{array}\right)$$

$$\overset{-1/3R_2\to R_2}{\rightsquigarrow}\left(\begin{array}{ccc|c}1 & 2 & 4 & 7\\ 0 & 1 & 2 & 5\\ 0 & 0 & 0 & 0\end{array}\right)\overset{R_1-2R_2}{\rightsquigarrow}\left(\begin{array}{ccc|c}1 & 0 & 0 & -3\\ 0 & 1 & 2 & 5\\ 0 & 0 & 0 & 0\end{array}\right)$$

Hence, z is free, $y=5-2z$ and $x=-3$ (**1 Point**) and the solution set is (**1 Point**)

$$S=\left\{\begin{pmatrix}-3\\ 5-2t\\ t\end{pmatrix}\mid t\in\mathbb{R}\right\}.$$

Version B.

Solve the linear system

$$\begin{array}{rcrcrcr} 4x & + & y & - & z & = & 1 \\ 3x & + & 2y & + & z & = & -2 \\ x & + & 4y & + & 5z & = & -8 \end{array}$$

Solution. We write the augmented matrix and use row reduction (**3 Points**)

$$\left(\begin{array}{ccc|c} 4 & 1 & -1 & 1 \\ 3 & 2 & 1 & -2 \\ 1 & 4 & 5 & -8 \end{array}\right) \overset{R_1 \leftrightarrow R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 4 & 5 & -8 \\ 3 & 2 & 1 & -2 \\ 4 & 1 & -1 & 1 \end{array}\right) \overset{R_2-3R_1\to R_2,\ R_3-4R_1\to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 4 & 5 & -8 \\ 0 & -10 & -14 & 22 \\ 0 & -15 & -21 & 33 \end{array}\right)$$

$$\overset{1/10R_2\to R_2}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 4 & 5 & -8 \\ 0 & 1 & 7/5 & -11/5 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{R_1-4R_2\to R_1}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & -3/5 & 4/5 \\ 0 & 1 & 7/5 & -11/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, z is free, $y = -11/5 - 7/5z$ and $x = 4/5 + 3/5z$ (**1 Point**) and the solution set is (**1 Point**)

$$S = \left\{ \begin{pmatrix} 4/5 + 3/5t \\ -11/5 - 7/5t \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Version C.

Solve the linear system

$$\begin{array}{rcrcrcr} 2x & - & y & + & z & = & -1 \\ 3x & + & 2y & - & z & = & 2 \\ x & + & 10y & - & 7z & = & 10 \end{array}$$

Solution. We write the augmented matrix and use row reduction (**3 Points**)

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 3 & 2 & -1 & 2 \\ 1 & 10 & -7 & 10 \end{array}\right) \overset{R_1 \leftrightarrow R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 10 & -7 & 10 \\ 3 & 2 & -1 & 2 \\ 2 & -1 & 1 & -1 \end{array}\right) \overset{R_2-3R_1\to R_2,\ R_3-2R_1\to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 10 & -7 & 10 \\ 0 & -28 & 20 & -28 \\ 0 & -21 & 15 & -21 \end{array}\right)$$

$$\overset{1/7R_2\leftrightarrow R_2}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 10 & -7 & 10 \\ 0 & 1 & -5/7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{R_1-10R_2\to R_1}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1/7 & 0 \\ 0 & 1 & -5/7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, z is free, $y = 1 + 5/7z$ and $x = -1/7z$ (**1 Point**) and the solution set is (**1 Point**)

$$S = \left\{ \begin{pmatrix} -1/7t \\ 1 + 5/7t \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Version D.

Solve the linear system

$$\begin{array}{rcrcrcr} 3x & + & 2y & + & z & = & 2 \\ 4x & - & y & + & 2z & = & 1 \\ x & + & 8y & - & z & = & 4 \end{array}$$

Solution. We write the augmented matrix and use row reduction (**3 Points**)

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ 4 & -1 & 2 & 1 \\ 1 & 8 & -1 & 4 \end{array}\right) \overset{R_1 \leftrightarrow R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 8 & -1 & 4 \\ 4 & -1 & 2 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right) \overset{R_2-4R_1\to R_2,\ R_3-3R_1\to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 8 & -1 & 4 \\ 0 & -33 & 6 & -15 \\ 0 & -22 & 4 & -10 \end{array}\right)$$

$$\overset{R_2/33\to R_2}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 8 & -1 & 4 \\ 0 & 1 & -2/11 & 5/11 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{R_1-8R_2\to R_1}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 5/11 & 4/11 \\ 0 & 1 & -2/11 & 5/11 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, z is free, $y = 5/11 + 2/11z$ and $x = 4/11 - 5/11z$ (**1 Point**) and the solution set is (**1 Point**)

$$S = \left\{ \begin{pmatrix} 4/11 - 5/11t \\ 5/11 + 2/11t \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Question 6. [7 Points]

Version A.

Consider the function of two variables $f(x, y) = \cos(\sqrt{x+y})$.

(a) Find the domain of f.

(b) Find the range of f.

(c) Sketch the domain of f in the plane $\mathbb{R}^2$.

(d) On a separate graph, sketch three level curves.

Solution. (a) We need $x + y \geq 0$ to take the square root, and hence $y \geq -x$. Hence, (**1 Point**)

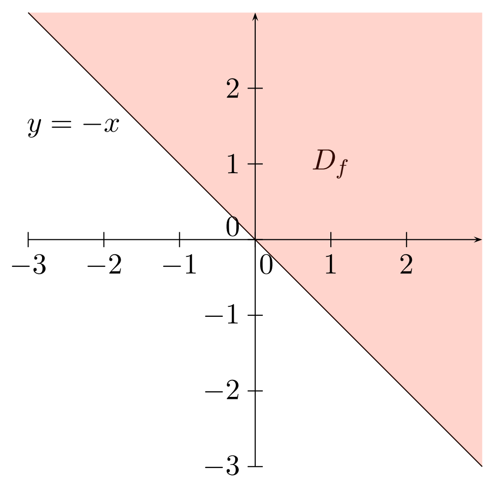
$$D_f = \{(x, y) \in \mathbb{R}^2 \mid y \geq -x\}.$$

(b) The range of f is $\cos(\mathbb{R}_+) = [-1, 1]$ (**1 Point**).

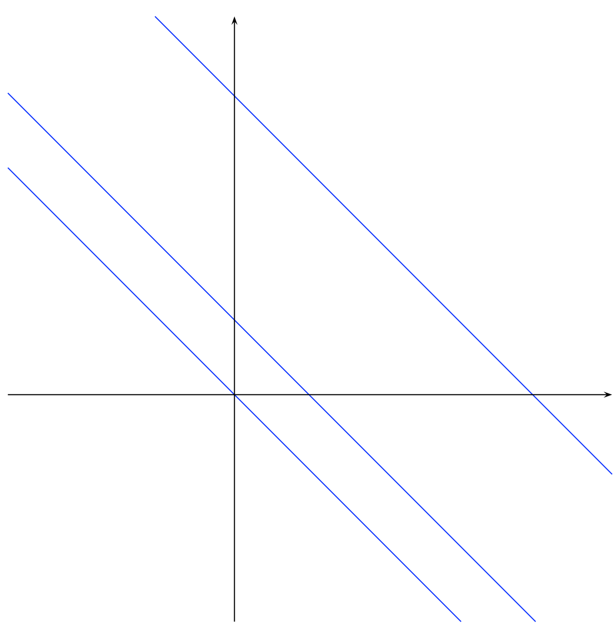
(c) (**1 Point**)

(d) We have

$$\begin{aligned} \cos(\sqrt{x+y}) &= c \\ \sqrt{x+y} &= \arccos(c) \\ y &= -x + [\arccos(c)]^2 \end{aligned}$$



$y=-x$
D_f
2
1
0
−3
−2
−1
0
1
2
−1
−2
−3



(**2 Points**). We graph these lines (**2 Points**).

Version B.

Consider the function of two variables $f(x,y) = \sin(\sqrt{x+y})$.

(a) Find the domain of f.

(b) Find the range of f.

(c) Sketch the domain of f in the plane $\mathbb{R}^2$.

(d) On a separate graph, sketch three level curves.

Solution. (a) We need $x + y \geq 0$ to take the square root, and hence $y \geq -x$. Hence, (**1 Point**)

$$D_f = \{(x,y) \in \mathbb{R}^2 \mid y \geq -x\}.$$

(b) The range of f is $\sin(\mathbb{R}_+) = [-1,1]$ (**1 Point**).

(c) (**1 Point**) [Same as Version A]

(d) We have

$$\begin{aligned} \sin(\sqrt{x+y}) &= c \\ \sqrt{x+y} &= \arcsin(c) \\ y &= -x + [\arcsin(c)]^2 \end{aligned}$$

(**2 Points**). We graph these lines [same as Version A] (**2 Points**)

Version C.

Consider the function of two variables $f(x,y) = \sin(\sqrt{x-y})$.

(a) Find the domain of f.

(b) Find the range of f.

(c) Sketch the domain of f in the plane $\mathbb{R}^2$.

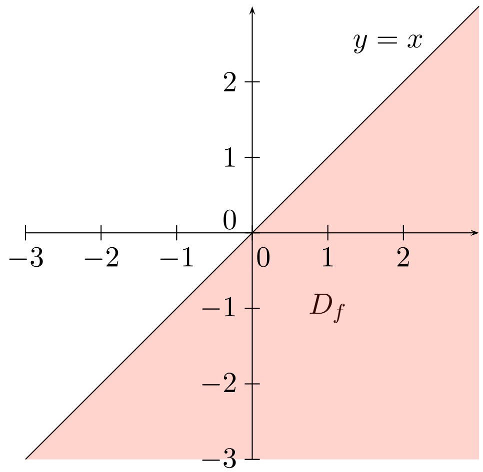
(d) On a separate graph, sketch three level curves.

Solution. (a) We need $x - y \geq 0$ to take the square root, and hence $y \leq x$. Hence, (**1 Point**)

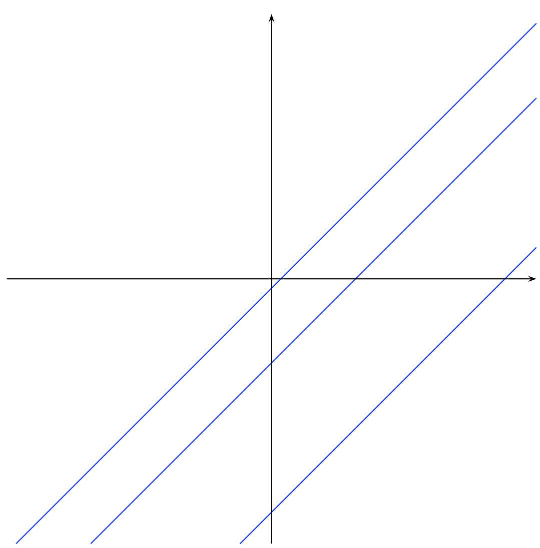
$$D_f = \{(x,y) \in \mathbb{R}^2 \mid y \leq x\}.$$

(b) The range of f is $\sin(\mathbb{R}_+) = [-1,1]$ (**1 Point**).

(c) (**1 Point**)



$y = x$
2
1
0
−3
−2
−1
0
1
2
D_f
−1
−2
−3



(d) We have

$$\begin{aligned} \sin(\sqrt{x-y}) &= c \\ \sqrt{x-y} &= \arcsin(c) \\ y &= x - [\arcsin(c)]^2 \end{aligned}$$

(**2 Points**). We graph these lines (**2 Points**).

Version D.

Question 6. [7 Points] Consider the function of two variables $f(x, y) = \cos(\sqrt{x-y})$.

(a) Find the domain of f.

(b) Find the range of f.

(c) Sketch the domain of f in the plane $\mathbb{R}^2$.

(d) On a separate graph, sketch three level curves.

Solution. (a) We need $x - y \geq 0$ in order to take the square root, and hence $y \leq x$. Hence, (**1 Point**)

$$D_f = \{(x, y) \in \mathbb{R}^2 \mid y \leq x\}.$$

(b) The range of f is $\cos(\mathbb{R}_+) = [-1, 1]$ (**1 Point**).
(c) (**1 Point**) [Same as Version C]
(d) We have

$$\begin{aligned} \cos(\sqrt{x-y}) &= c \\ \sqrt{x-y} &= \arccos(c) \\ y &= x - [\arccos(c)]^2 \end{aligned}$$

(**2 Points**). We graph these lines (**2 Points**).
[Same as Version C]

Bonus. [4 Points]

Version A .

Solve the following equation:

$$z^3 = -1 + i.$$

Hint: write $z = re^{i\theta}$ and $-1 + i$ in polar form

Solution. We calculate $|-1+i| = \sqrt{2}$ and $\arctan(-1) = -\pi/4$. Thus, since x-coordinate is < 0, $-1 + i = \sqrt{2}e^{i(3\pi/4)}$ (**1 Point**). Writing $z = re^{i\theta}$, the equation becomes (**1 Point**)

$$r^3 z^{i(3\theta)} = \sqrt{2}e^{i(3\pi/4)},$$

that is

$$\begin{cases} 3\theta &= 3\pi/4 + 2k\pi \\ r^3 &= \sqrt{2} \end{cases} \Rightarrow \begin{cases} \theta &= \pi/4 + k2\pi/3 \\ r &= \sqrt[6]{2} \end{cases}$$

where $k = 0, 1, 2$ (**1 Point**). Thus, since $\pi/4 + 2\pi/3 = 11\pi/12$ and $\pi/4 + 4\pi/3 = 19\pi/12$, the solutions are (**1 Point**)

$$\sqrt[6]{2}e^{i(\pi/4)},\ \sqrt[6]{2}e^{i(11\pi/12)} \text{ and } \sqrt[6]{2}e^{i(19\pi/12)}.$$

Version B.

Solve the following equation:

$$z^3 = -1 - i.$$

Hint: write $z = re^{i\theta}$ and $-1 - i$ in polar form

Solution. We calculate $|-1-i| = \sqrt{2}$ and $\arctan(1) = \pi/4$. Thus, since the x-coordinate is < 0, $-1 - i = \sqrt{2}e^{-i(3\pi/4)}$ (**1 Point**). Writing $z = re^{i\theta}$, the equation becomes (**1 Point**)

$$r^3 z^{i(3\theta)} = \sqrt{2}e^{-i(3\pi/4)},$$

that is

$$\begin{cases} 3\theta &= -3\pi/4 + 2k\pi \\ r^3 &= \sqrt{2} \end{cases} \Rightarrow \begin{cases} \theta &= -\pi/4 + k2\pi/3 \\ r &= \sqrt[6]{2} \end{cases}$$

where $k = 0, 1, 2$ (**1 Point**). Thus, since $-\pi/4 + 2\pi/3 = (5/12)\pi$ and $-\pi/4 + 4\pi/3 = (13/12)\pi$, the solutions are (**1 Point**)

$$\sqrt[6]{2}e^{-i(\pi/4)},\ \sqrt[6]{2}e^{i(5/12\pi)} \text{ and } \sqrt[6]{2}e^{i(13/12\pi)}.$$

Version C.

Solve the following equation:

$$z^3 = 1 - i.$$

Hint: write $z = re^{i\theta}$ and $1 - i$ in polar form.

Solution. We calculate $|1 - i| = \sqrt{2}$ and $\arctan(-1) = -\pi/4$. Hence $1 + i = \sqrt{2}e^{-i\pi/4}$ (**1 Point**). Writing $z = re^{i\theta}$, the equation becomes (**1 Point**)

$$r^3 z^{i(3\theta)} = \sqrt{2}e^{-i(\pi/4)},$$

that is

$$\begin{cases} 3\theta &= -\pi/4 + 2k\pi \\ r^3 &= \sqrt{2} \end{cases} \Rightarrow \begin{cases} \theta &= -\pi/12 + k2\pi/3 \\ r &= \sqrt[6]{2} \end{cases}$$

where

$$\sqrt[6]{2}e^{-i(\pi/12)},\ \sqrt[6]{2}e^{i(7\pi/12)} \text{ and } \sqrt[6]{2}e^{i(15\pi/12)}.$$

Version D.

Solve the following equation:

$$z^3 = 1 + i.$$

Hint: write $z = re^{i\theta}$ and $1 + i$ in polar form

Solution. We calculate $|1 + i| = \sqrt{2}$ and $\arctan(1) = \pi/4$. So $1 + i = \sqrt{2}e^{i(\pi/4)}$ (**1 Point**). Writing $z = re^{i\theta}$, the equation becomes (**1 Point**)

$$r^3 z^{i3\theta} = \sqrt{2}e^{i(\pi/4)}.$$

Thus

$$\begin{cases} 3\theta & = & \pi/4 + 2k\pi \\ r^3 & = & \sqrt{2} \end{cases} \Rightarrow \begin{cases} \theta & = & \pi/12 + k2\pi/3 \\ r & = & \sqrt[6]{2} \end{cases}$$

where$k = 0, 1, 2$ (**1 Point**). Thus, since $\pi/12 + 2\pi/3 = 3\pi/4$ and $\pi/12 + 4\pi/3 = 17\pi/12$, the solutions are (**1 Point**)

$$\sqrt[6]{2}e^{i(\pi/12)}, \ \sqrt[6]{2}e^{i(3\pi/4)} \text{ et } \sqrt[6]{2}e^{i(17\pi/12)}.$$