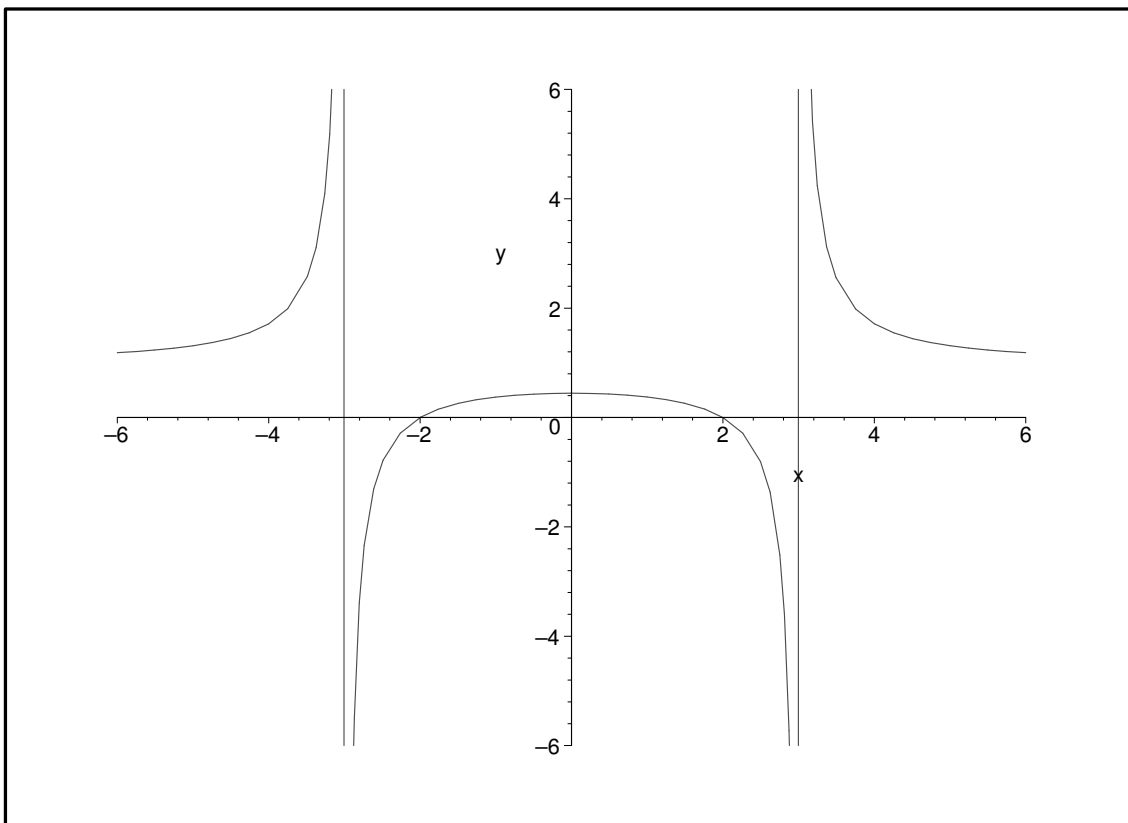


CARLETON UNIVERSITY
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1009C Calculus for Business and Economics

Notes on Curve Sketching

These notes are intended to supplement Section 2.4 of the textbook.

Suppose we are given a function such as $f(x) = \frac{x^2 - 4}{x^2 - 9}$. A natural question to ask is: what does it look like? (More specifically, what does the graph of the function look like?) If we have a graphical calculator, or a computer program with similar abilities, we could program it and ask it to tell us the answer. If we did so, we'd get something such as the following:



But what if we had to work things out for ourselves? We would need to know:

- where the function is increasing or decreasing;
- where it is concave up or down;
- if it has any local maxima or minima, or points of inflection;
- if there are any horizontal or vertical asymptotes;
- where it intersects the x and y -axes.

One useful thing we'll need will be the familiar formula for finding solutions of quadratic equations.

The Quadratic Formula. *The quadratic equation $ax^2 + bx + c = 0$ has solutions*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In what follows, we'll describe our strategy for how to draw sketches of graphs of functions, then afterwards we'll work through some examples.

Strategy

1. Find the first and second derivatives, $f'(x)$ and $f''(x)$, of the function $f(x)$. (This may require us to use rules such as the quotient rule and chain rule).
2. Make a list of all the values of x where something interesting might happen. By this, we mean the following:
 - (a) values of x where $f(x)$ is not defined¹;
 - (b) critical values (i.e. where the derivative $f'(x) = 0$);
 - (c) possible inflection points (i.e. where the second derivative $f''(x) = 0$).
3. Put these values in a table. Now, these will divide the “real line”, i.e. $(-\infty, \infty)$, into intervals (from $-\infty$ to the first interesting point, then between the first and second point, and so on). In each interval, we choose a “test point”: it doesn't matter which values we choose, however to make our lives easier we should choose something straightforward.

At each test point, we evaluate $f'(a)$ and apply the First Derivative Rule to determine whether $f(x)$ is increasing or decreasing on that interval. We then evaluate $f''(a)$ and apply the Second Derivative Rule to determine whether the function is concave up or concave down on that interval.

After that, each critical value $x = c$, we apply the First Derivative Test to see if that critical value is a local maximum or minimum. If, as we pass from the left of $x = c$ to right of it, the value of $f(x)$ changes from positive to negative, then we have a local maximum at co-ordinates $(c, f(c))$; if it changes from negative to positive, then we have a local minimum at co-ordinates $(c, f(c))$. If the sign of the derivative doesn't change, then we have neither. If we didn't have any critical values at all, then we have nothing to do at all and then move on to the next step.

Then we do essentially the same thing to find the points of inflection. For each possible value where there may be a point of inflection, say $x = b$, we do the following. If the concavity changes (from concave up to concave down, or vice-versa) as we pass $x = b$, then we must have a point of inflection at co-ordinates $(b, f(b))$. If the concavity doesn't change, then there is not a point of inflection at $x = b$, while if we have no candidate values, we can omit this step completely.

¹Beware that some books include such values under the heading of critical points.

4. Now we locate the vertical and horizontal asymptotes.

- (a) If the function $f(x)$ has the vertical line $x = a$ as a vertical asymptote, then $x = a$ must be one of the values for which $f(x)$ is not defined. So we need to examine what happens as x approaches a from each side. In other words, we need to consider the limits $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$. If at least one of these limits is either ∞ or $-\infty$, then the line $x = a$ is a vertical asymptote of $f(x)$.

However, if the function is a rational function, i.e. we have $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, then there is an easier way (which avoids having to actually evaluate any limits). We have the following test:

The Vertical Asymptote Test. *The line $x = a$ is a vertical asymptote of the rational function $f(x) = \frac{P(x)}{Q(x)}$ if $Q(a) = 0$ but $P(a) \neq 0$.*

So in this case, we just have to evaluate the two polynomials $P(x)$ and $Q(x)$ in order to determine if we have a vertical asymptote.

- (b) To find if $f(x)$ has any horizontal asymptotes, we need to examine the limits $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$. If one of these limits is a number, say ℓ , then the line $y = \ell$ must be a horizontal asymptote of $f(x)$. (This means that a function has at most two possible horizontal asymptotes, whereas in theory a function can have any number of vertical asymptotes.)

5. Now we find the co-ordinates of the points where the function intersects with the y -axis and the x -axis. Now, provided that the function is defined at $x = 0$, then we just have to evaluate $f(0)$ and we know that the y -intercept has co-ordinates $(0, f(0))$. (If the function is not defined at $x = 0$, then there can't be a y -intercept.)

To find the points where the function crosses the x -axis (if there are any such points), we have to solve the equation $f(x) = 0$ for x : any solution to this, say $x = a$, will tell us that there is an x -intercept with co-ordinates $(a, 0)$. If the equation has no solutions, then there are no x -intercepts.

6. Having carried out all the steps listed above, we are now ready to sketch the graph. First we mark the co-ordinates of any axis-intercepts, local maxima or minima, inflection points, or points where the function is not defined. Then we draw the vertical and horizontal asymptotes. Finally, using our knowledge of where the function is increasing or decreasing, as well as its concavity, to draw in the graph of the function itself.

A worked example

Let's work through the example we saw before, namely the function $f(x) = \frac{x^2 - 4}{x^2 - 9}$.

1. First, we need to find the first and second derivatives of our function. So we need to use the quotient rule:

$$\begin{aligned}f'(x) &= \frac{2x(x^2 - 4) - 2x(x^2 - 9)}{(x^2 - 9)^2} \\ &= \frac{-10x}{(x^2 - 9)^2}.\end{aligned}$$

To find the second derivative, as well as the quotient rule, we'll also need the generalised power rule (to find the derivative of the denominator):

$$\begin{aligned}f''(x) &= \frac{-10(x^2 - 9)^2 - (-10x) \cdot 2 \cdot (x^2 - 9) \cdot 2x}{(x^2 - 9)^4} \\ &= \frac{-10(x^2 - 9) + 40x^2}{(x^2 - 9)^3} \\ &= \frac{30x^2 + 90}{(x^2 - 9)^3}.\end{aligned}$$

Note that we simplified each of these as much as we could: this was (i) to make computing the second derivative easier, and (ii) when we have to evaluate these derivatives later on, we want this to be straightforward.

2. Now we want to find all the “interesting” values of x for this function.
 - (a) First, we find the values of x for which $f(x)$ is not defined. Since $f(x)$ is a rational function, it is not defined precisely when the denominator is zero. Now, we have

$$\begin{aligned}x^2 - 9 &= 0 \\ \Leftrightarrow x^2 &= 9 \\ \Leftrightarrow x &= \pm 3\end{aligned}$$

so there are two values, $x = 3$ and $x = -3$, where $f(x)$ is not defined.

- (b) To find the critical values, we look at the first derivative $f'(x)$ and see where it equals 0. So we have:

$$\begin{aligned}f'(x) &= 0 \\ \Leftrightarrow \frac{-10x}{(x^2 - 9)^2} &= 0 \\ \Leftrightarrow -10x &= 0 \\ \Leftrightarrow x &= 0.\end{aligned}$$

Thus our function has exactly one critical point, at $x = 0$.

- (c) To find the possible inflection points, we look at the second derivative $f''(x)$ and see where that equals 0. This time, we have:

$$\begin{aligned} f''(x) &= 0 \\ \Leftrightarrow \frac{30x^2 + 90}{(x^2 - 9)^3} &= 0 \\ \Leftrightarrow 30x^2 + 90 &= 0 \\ \Leftrightarrow x^2 + 3 &= 0 \\ \Leftrightarrow x^2 &= -3, \end{aligned}$$

which has no real number as a solution. Thus this function $f(x)$ cannot have any inflection points.

3. Now we take our answers to part 2 and put them in a table. Then in each interval, we choose some test points; I've chosen -4 , -1 , 1 and 4 , but you could have chosen some others if you wanted. At each point, we evaluate $f'(x)$ and $f''(x)$ and add these to our table; if the answer does not exist, we write *DNE*.

x	$f'(x)$	$f''(x)$	Behaviour
-4	$\frac{40}{49}$	$\frac{570}{343}$	Increasing Concave up
-3	<i>DNE</i>	<i>DNE</i>	
-1	$\frac{10}{64}$	$-\frac{120}{512}$	Increasing Concave down
0	0	$-\frac{10}{81}$	Local maximum Concave down
1	$-\frac{10}{64}$	$-\frac{120}{512}$	Decreasing Concave down
3	<i>DNE</i>	<i>DNE</i>	
4	$-\frac{40}{49}$	$\frac{570}{343}$	Decreasing Concave up

4. Our next step is to find any asymptotes of $f(x)$.

- (a) As $f(x) = \frac{x^2 - 4}{x^2 - 9}$ is a rational function, we can use the Vertical Asymptote Test. The two values we have to try are those where the denominator of $f(x)$ is zero, namely $x = 3$ and $x = -3$:

- at $x = 3$, we have $3^2 - 9 = 0$ but $3^2 - 4 = 5 \neq 0$;
- at $x = -3$, we have $(-3)^2 - 9 = 0$ but $(-3)^2 - 4 = 5 \neq 0$.

So both $x = 3$ and $x = -3$ are vertical asymptotes of $f(x)$.

(b) To determine if $f(x)$ has any horizontal asymptotes, we check the limits $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$. So we have:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x^2 - 4}{x^2 - 9} &= \lim_{x \rightarrow \infty} \frac{1 - \frac{4}{x^2}}{1 - \frac{9}{x^2}} \\ &= \frac{1 - 0}{1 - 0} \\ &= 1,\end{aligned}$$

and by the same calculation, $\lim_{x \rightarrow -\infty} \frac{x^2 - 4}{x^2 - 9} = 1$. So we have one horizontal asymptote, namely the line $y = 1$.

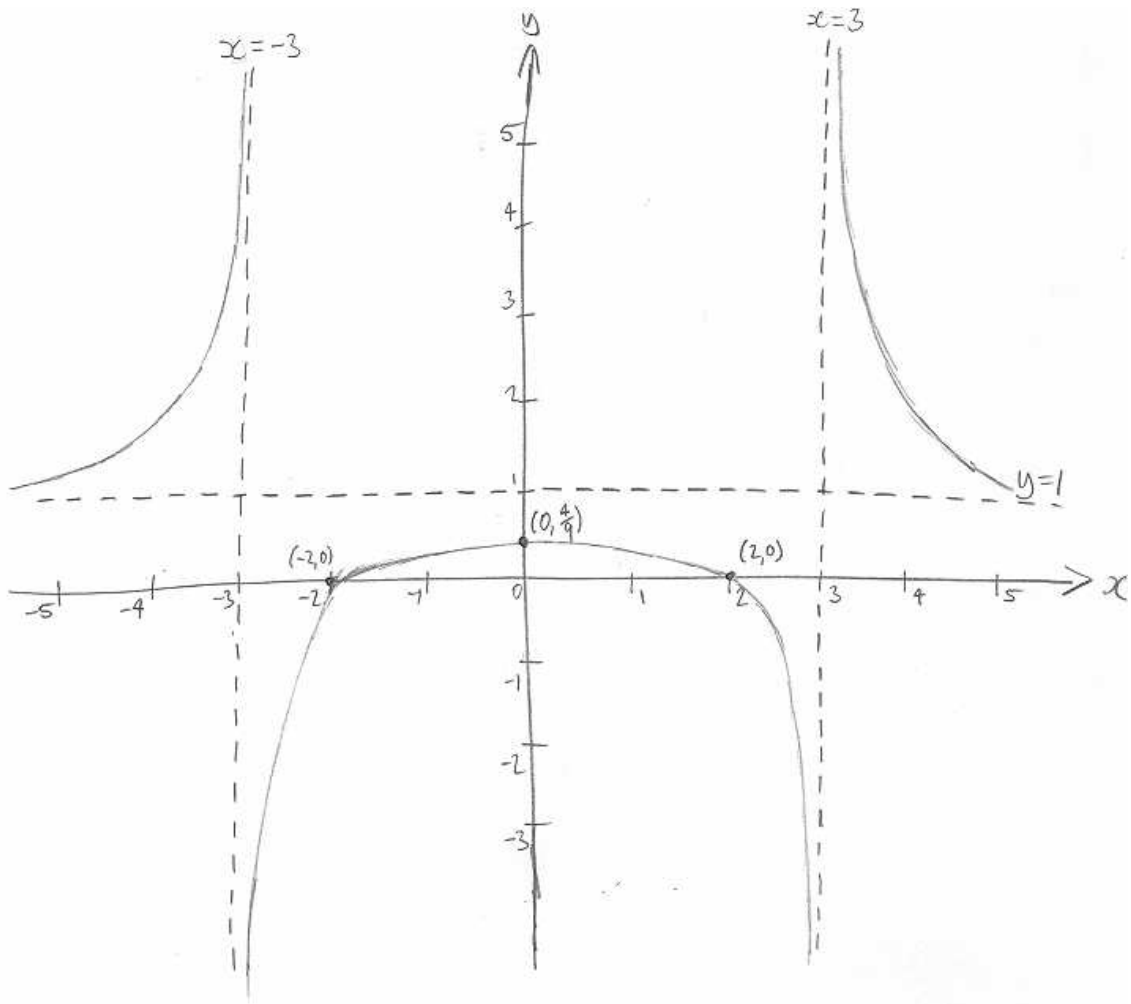
5. Now we find the co-ordinates of the x - and y -axis intercepts. To find the y -intercept, we just evaluate $f(0) = \frac{0^2 - 4}{0^2 - 9} = \frac{4}{9}$, so $f(x)$ intersects the y -axis at co-ordinates $\left(0, \frac{4}{9}\right)$.

To find the x -axis intercepts, we solve the equation $f(x) = 0$ for x . So we have:

$$\begin{aligned}\frac{x^2 - 4}{x^2 - 9} &= 0 \\ \Leftrightarrow x^2 - 4 &= 0 \\ \Leftrightarrow x^2 &= 4 \\ \Leftrightarrow x &= \pm 2.\end{aligned}$$

Hence $f(x)$ intersects the x -axis at co-ordinates $(-2, 0)$ and $(2, 0)$.

6. Now we have all the information we need to draw a sketch of the graph of $f(x)$. We start by marking all points of interest on the graph, namely x - and y -axis intercepts and our local maximum. (If we had any points of inflection, we'd mark these too.) Now, there is a local maximum when $x = 0$, so its co-ordinates are $(0, f(0)) = \left(0, \frac{4}{9}\right)$. Then we draw in the horizontal and vertical asymptotes. Finally, we use our knowledge of where $f(x)$ is increasing and decreasing, as well as its concavity, to draw the curve through the specified points with the correct shape. So we should get something like the graph on the next page.



Some exercises

We finish with some exercises for you to try, in addition to the homework exercises on this topic. For each function $f(x)$, sketch its graph:

1. $f(x) = 1 - x - x^3$;
2. $f(x) = \frac{x}{x^2 + 1}$;
3. $f(x) = \frac{x+1}{2x-1}$;
4. $f(x) = \frac{1}{2}x^2 + \frac{1}{x}$;
5. $f(x) = \sqrt{x^2 + 4}$.