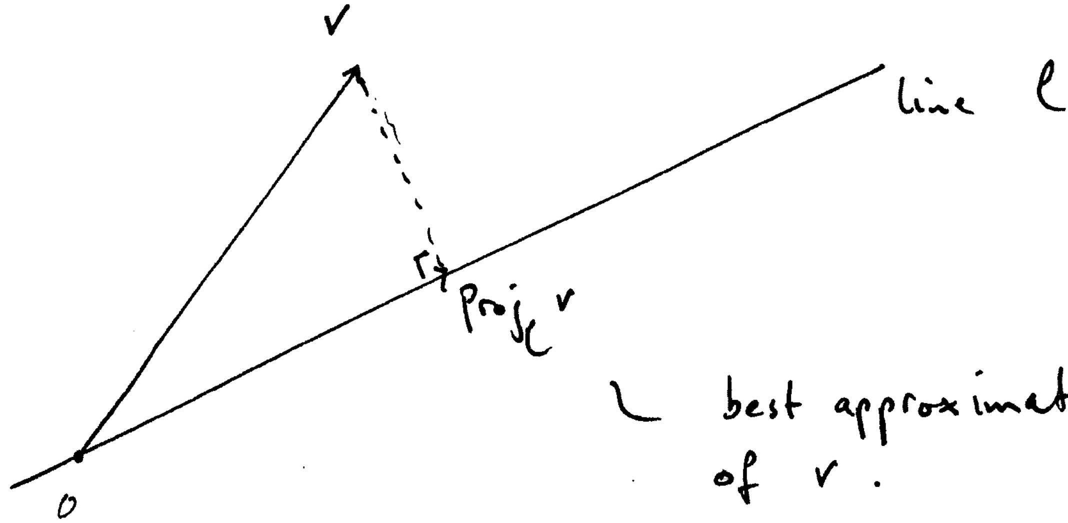
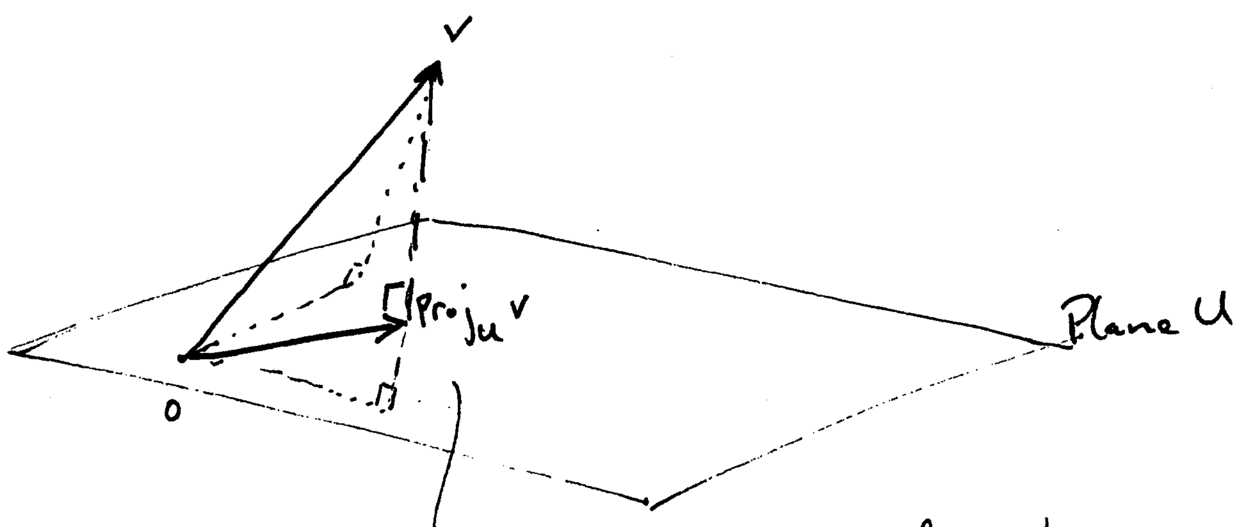
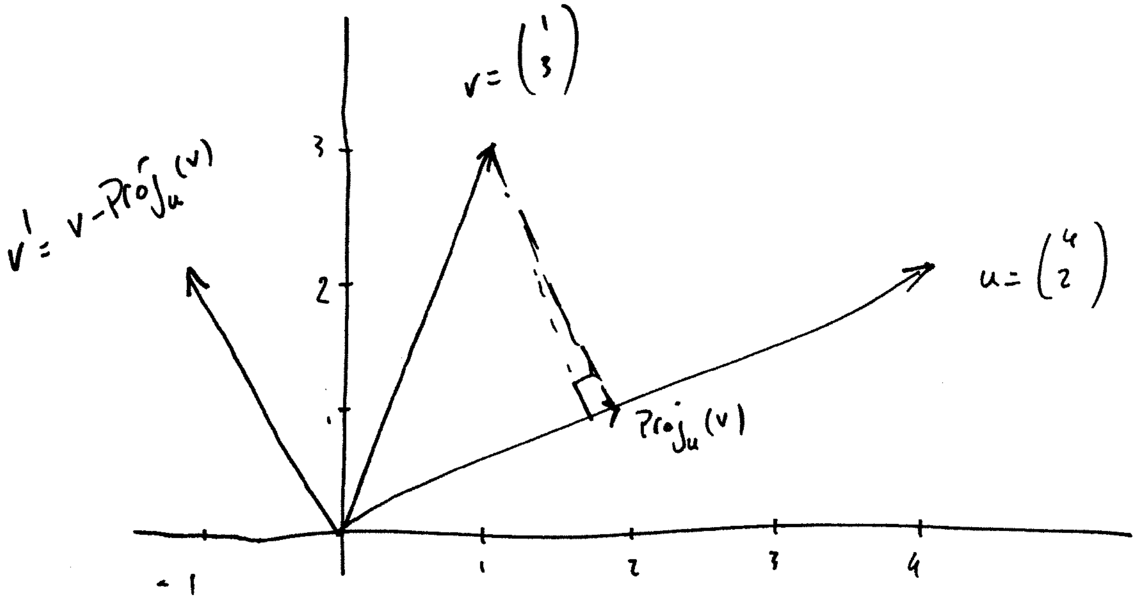
Projection & Best Approximation



best approximation in ℓ of v.



best approximation of v by vector in U.



$$\mathrm{Proj}_u(v) = \frac{v\cdot u}{u\cdot u}\cdot u = \frac{1\cdot 4 + 3\cdot 2}{4\cdot 4 + 2\cdot 2}\begin{pmatrix}4\\2\end{pmatrix} = \frac{10}{20}\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}.$$

Then $v' = v - \mathrm{Proj}_u(v) = \begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$

and $\{u, v'\} = \left\{\begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^2$.

If we want, we can also normalize: replace u by $\frac{u}{\|u\|}$ and v' by $\frac{v'}{\|v'\|}$: this gives an orthonormal basis

$$\left\{\frac{u}{\|u\|}, \frac{v'}{\|v'\|}\right\} = \left\{\begin{pmatrix}\frac{2\sqrt{5}}{5}\\ \frac{\sqrt{5}}{5}\end{pmatrix}, \begin{pmatrix}-\frac{\sqrt{5}}{5}\\ \frac{2\sqrt{5}}{5}\end{pmatrix}\right\}.$$

Orthogonal Bases and Gram-Schmidt.

Recall the standard basis for $\mathbb{R}^n$:

$$B = \left\{ \underset{e_1}{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}}, \underset{e_2}{\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}}, \cdots, \underset{e_n}{\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}} \right\}.$$

This basis has two important properties:

① all basis vectors are pairwise orthogonal:

$$\langle e_i, e_j \rangle = 0 \quad \text{if} \quad i \neq j$$

② $\|e_i\| = 1$: all basis vectors have unit length.

We say that the standard basis is <u>orthonormal</u>.

The following basis of $\mathbb{R}^2$ is not orthonormal

$$B = \left\{ \begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \right\}$$ (neither orthogonal nor normalized).

What could we do to make it orthogonal?

<u>Sol</u>^{n} Use <u>projections</u>.

Gram-Schmidt : general case

Let $\mathcal{B} = \{v_1, \dots, v_k\}$ be a basis for $U \subseteq \mathbb{R}^n$.

Set
$$u_1 = v_1$$
$$u_2 = v_2 - \mathrm{Proj}_{u_1} v_2$$
$$u_3 = v_3 - \mathrm{Proj}_{u_1} v_3 - \mathrm{Proj}_{u_2} v_3$$
$$\vdots$$
$$u_k = v_k - \mathrm{Proj}_{u_1} v_k - \cdots - \mathrm{Proj}_{u_{k-1}} v_k$$

Then $\{u_1, \dots, u_k\}$ is an orthogonal basis for U.

(If you want to have an orthonormal basis, replace by
$$\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \dots, \frac{u_k}{\|u_k\|} \right\}.$$
)

Gram-Schmidt Example

Lecture 21

Let $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 1 \end{pmatrix}$, and let

$U = \text{span}\{v_1, v_2, v_3\}$.

Note: $\{v_1, v_2, v_3\}$ is a basis for U, but not orthogonal.

Set
- $u_1 = v_1$
- $u_2 = v_2 - \dfrac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1 \\ -1/2 \end{pmatrix}$

$\rightsquigarrow$ replace by $u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}$ for easier arithmetic..

- $u_3 = v_3 - \dfrac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 - \dfrac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2$

$$= \begin{pmatrix} 0 \\ 3 \\ -1 \\ 1 \end{pmatrix} - \frac{4}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{6}\begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Then $\{u_1, u_2, u_3\}$ is an orthogonal basis for U.

Let $w = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$; find $\text{Proj}_U(w)$.

Sol^n $\text{Proj}_U(w) = \dfrac{u_1 \cdot w}{u_1 \cdot u_1} u_1 + \dfrac{u_2 \cdot w}{u_2 \cdot u_2} \cdot u_2 + \dfrac{u_3 \cdot w}{u_3 \cdot u_3} u_3$

$$= \frac{6}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{4}{6}\begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix} + \frac{-5}{3}\begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

In general, finding such a projection / best approximation can be difficult. But if we have an orthogonal basis for U, it is easy.

<u>Def</u> Let $B = \{v_1, \dots, v_k\}$ be an orthogonal basis for $U \subseteq \mathbb{R}^n$, and let $v \in \mathbb{R}^n$.

Then $\boxed{\operatorname{proj}_U v = \operatorname{proj}_{v_1} v + \operatorname{proj}_{v_2} v + \cdots + \operatorname{proj}_{v_k} v}$

is the <u>projection</u> of v onto U.

$\operatorname{Proj}_U v$ is the best approximation to v by a vector in U.

<u>Ex</u> $B = \left\{ \underset{v_1}{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}} \right\}$

is an orth. basis for $\operatorname{Span}(B) = U$.

Find the best approximation by a vector in U to $v = (3, 2, 1, 0)$.

$$\operatorname{Proj}_U v = \frac{v \cdot v_1}{v_1 \cdot v_1} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{v \cdot v_2}{v_2 \cdot v_2} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{v \cdot v_3}{v_3 \cdot v_3} \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}$$

$$= \frac{3}{3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{7}{6} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{5}{9} \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}$$