University of Calgary - Faculty of Science

Department of Mathematics and Statistics

AMAT 219 Midterm Test - Winter 2011

Closed Book , No Calculators.

01. Find $\int_1^{e^2} \frac{\ln(x)}{\sqrt{x}}\, dx$

02. Evaluate $\int 2x \sin^{-1}(x^2)\, dx$.

03. Determine $\int \frac{3x^2 - 2x + 12}{x^2 + 4}\, dx$.

04. Compute $\int \frac{1}{\sqrt{x^2 - 4x + 5}}\, dx$.

05. Find $\int \frac{4}{x^3 - 2x^2}\, dx$.

06. Evaluate $\int \frac{9}{x^2\sqrt{x^2 - 9}}\, dx$, $x > 3$.

07. Let $\mathbf{J} = \int \frac{1}{x(4x^2+1)^2}\, dx$. The substitution $x = \frac{1}{2}\tan(\theta)$ transforms the integral $\mathbf{J}$ into $\int g(\theta)\, d\theta$, find $g(\theta)$.

08. Determine whether the improper integral $\int_1^{\infty} \frac{d}{dx}\left\{\frac{6x + \ln(x)}{2x+1}\right\} dx$ converges or diverges and find its value if it exists.

09. Determine whether the improper integral $\int_0^2 \frac{1}{(x-1)^2}\, dx$ converges or diverges and find its value if it exists.

10. Find the **Simpson's Rule** approximation $\mathbf{S}_{10}$ for $\int_0^1 (2x^3 - x)\, dx$.

11. Let f be a function such that $-6 \le f''(x) \le 2$, for $x \in [0,2]$ and let $\mathbf{J} = \int_0^2 f(x)\, dx$.

The **Trapezoid Rule $\mathbf{T}_n$** is used to estimate the value of the integral $\mathbf{J}$ with an absolute error

0.04. Determine the smallest number of subintervals n.

12. Express the iterated integral $\mathbf{J} = \int_0^1 \left(\int_{-3x}^{x} f(x,y)\, dy\right) dx$ as an equivalent iterated integral in which

the $x-$*integration* is performed first and the $y-$*integration* is performed second.

13. Find he volume of the solid enclosed by the surfaces $z = 3\sqrt{x^2 + y^2}$ and $z = 6 - 3(x^2 + y^2)$.

14. Let $\mathbf{D}$ be the region in the $xy-$*plane* enclosed by the Trapezoid with vertices at the points

$(0,0)$, $(2,4)$, $(5,4)$ and $(6,0)$. Determine the value of $\iint_{\mathbf{D}} dA$

15. A thin plate has the shape of the planar region enclosed by the parabola $x = y^2$ and the straight lines

$x = 0$ and $y = 1$. Find the $y-$coordinate of the centroid of the plate.

16. A thin plate has the shape of the planar region described by $0 \le y \le \sin(2x)$, $0 \le x \le \frac{\pi}{4}$.

If the density function $\delta(x,y) = 8x$, find the mass m of the plate.

Solutions to AMAT 219 Midterm

1. $\int_1^{e^2} \frac{\ln(x)}{\sqrt{x}}\, dx \leftarrow$ By parts Once.

$$= 2x^{\frac12}\ln(x) - \int \frac{1}{x}\cdot 2x^{\frac12}\, dx$$

$$= 2x^{\frac12}\ln(x) - 2\int x^{-\frac12}\, dx$$

$$= 2x^{\frac12}\ln(x) - 4x^{\frac12}$$

$$= 2\sqrt{x}\left[\ln(x) - 2\right]\Big|_{x=1}^{x=e^2}$$

$$\begin{tikzcd} \ln(x) & \frac{1}{\sqrt{x}} = x^{-\frac12} \\ \frac{1}{x} & 2x^{\frac12} \arrow[l, "-\int"'] \arrow[ul] \end{tikzcd}$$

$$= 2\sqrt{e^2}\left[\ln(e^2) - 2\right] - 2\sqrt{1}\left[\ln(1) - 2\right]$$

$$= 2e[2-2] - 2(0-2) = 0 - 2(-2) = 4$$

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2. $I = \int 2x \sin^{-1}(x^2)\, dx \leftarrow$ substitution

Let $t = x^2$, $dt = 2x\,dx$

$I = \int \sin^{-1}(x^2) .. 2x\,dx = \int \sin^{-1}(t)\, dt \leftarrow$ By parts once.

$$I = t\sin^{-1}(t) - \int \frac{t}{\sqrt{1-t^2}}\, dt \;\cdots\; (*)$$

$$\begin{tikzcd} \sin^{-1}(t) & 1 \\ \frac{1}{\sqrt{1-t^2}} & t \arrow[l, "-\int"'] \arrow[ul] \end{tikzcd}$$

Consider $J = \int \frac{t}{\sqrt{1-t^2}}\, dt \leftarrow$ substitution

Let $u = 1 - t^2$, $du = -2t\,dt$

$\Rightarrow t\,dt = -\frac12\, du$

$$J = -\frac12 \int \frac{1}{\sqrt{u}}\, du = -\frac12 \int u^{-\frac12}\, du = -u^{\frac12} = -\sqrt{u} = -\sqrt{1-t^2}$$

Therefore $(*) \Rightarrow I = t\sin^{-1}(t) + \sqrt{1-t^2} + C$.

But $t = x^2$, hence $I = x^2\sin^{-1}(x^2) + \sqrt{1 - x^4} + C$

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3. $\displaystyle\int \frac{3x^2 - 2x + 12}{x^2+4}\,dx$

The integrand is an improper rational function.
Use long division first!

$$I = \int \left(3 - \frac{2x}{x^2+4}\right) dx \qquad \begin{array}{r|l} & 3 \leftarrow \text{quotient} \\ \hline x^2+4 & 3x^2 - 2x + 12 \\ & 3x^2 \qquad\quad + 12 \\ \hline & -2x \end{array}$$

$$= 3x - \ln(x^2+4) + C$$

$\uparrow$ Remainder

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4. $\displaystyle I = \int \frac{1}{\sqrt{x^2-4x+5}}$ $\leftarrow$ Complete the Square.

Recall: $Ax^2+Bx+C = \left(\sqrt{A}\,x + \frac{B}{2\sqrt{A}}\right)^2 + \left(C - \frac{B^2}{4A}\right), \; A > 0$

$$\therefore \; x^2-4x+5 = \left(\sqrt{1}\,x + \frac{-4}{2\sqrt{1}}\right)^2 + \left(5 - \frac{(-4)^2}{4(1)}\right)$$

$$= (x-2)^2 + 1$$

$$\therefore \; I = \int \frac{1}{\sqrt{(x-2)^2+1}}\,dx$$

Let $t = x-2$, hence $dt = dx$

$$\therefore \; I = \int \frac{1}{\sqrt{t^2+1}}\,dt \overset{\text{TABLE}}{=} \sinh^{-1}\left(\frac{t}{1}\right) + C$$

$$= \sinh^{-1}(x-2) + C$$

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5. $I = \int \frac{4}{x^3 - 2x^2}\,dx \leftarrow$ partial Fraction Decomposition

Consider $\frac{4}{x^3-2x^2} = \frac{4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$

Multiplying both sides by $x^2(x-2)$, we get

$$4 = Ax(x-2) + B(x-2) + Cx^2$$

At $x=0$: $4 = 0 + B(0-2) + 0$

$$\Rightarrow 4 = -2B \Rightarrow \boxed{B=-2}$$

At $x=2$: $4 = 0 + 0 + C.2^2$

$$\Rightarrow 4 = 4C \Rightarrow \boxed{C=1}$$

Comparing Coefficients of x^2 in both sides:

$$0 = A + C$$

But $C=1$, hence $0 = A+1 \Rightarrow \boxed{A=-1}$

$$\therefore \frac{4}{x^2(x-2)} = -\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x-2}$$

$$I = -\int \frac{1}{x}\,dx - 2\int x^{-2}\,dx + \int \frac{1}{x-2}\,dx$$

$$= -\ln|x| + \frac{2}{x} + \ln|x-2| + C$$

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6. $I = \int \frac{9}{x^2\sqrt{x^2-9}}\,dx$, $x > 3$

Integrand Contains the special quantity

$$b^2x^2 - a^2 = x^2 - 9 \quad (\text{with } a=3,\ b=1),$$

Therefore we let $x = \frac{a}{b}\sec(\theta)$

$\Rightarrow x = 3\sec(\theta)$ ------ (1)

$\therefore dx = 3\sec(\theta)\tan(\theta)\,d\theta$ .... (2)

and that $x^2 - 9 = (3\sec(\theta))^2 - 9 = 9(\sec^2(\theta) - 1)$

$\Rightarrow x^2 - 9 = 9\tan^2(\theta)$ ----- (3)

Substituting (1), (2), and (3) into I, we get

$$I = \int \frac{9}{(3\sec(\theta))^2\sqrt{9\tan^2(\theta)}} \cdot 3\sec(\theta)\tan(\theta)\,d\theta$$

$$= \int \frac{\cancel{9}\cdot\cancel{3}\sec(\theta)\cancel{\tan(\theta)}}{\cancel{9}\sec^2(\theta)\cdot\cancel{3}\cancel{\tan(\theta)}}\,d\theta$$

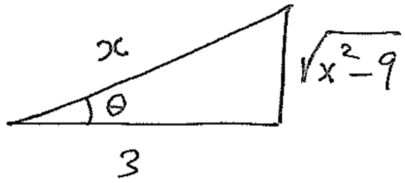
$$= \int \frac{1}{\sec(\theta)}\,d\theta = \int \cos(\theta)\,d\theta$$

$$= \sin(\theta) + C$$

$$\underset{\Delta}{\overset{\text{From}}{=}} \frac{\sqrt{x^2-9}}{x} + C$$

$x = 3\sec(\theta)$

$\Rightarrow \sec(\theta) = \frac{x}{3}$



$\sin(\theta) = \frac{\sqrt{x^2-9}}{x}$

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7. $J = \int \frac{1}{x(4x^2+1)^2}\,dx$

Let $x = \frac{1}{2}\tan(\theta)$ ... (1)

$dx = \frac{1}{2}\sec^2(\theta)\,d\theta$ ... (2), and

$4x^2 + 1 = 4\left(\frac{1}{2}\tan(\theta)\right)^2 + 1 = \tan^2(\theta) + 1$

$\therefore 4x^2 + 1 = \sec^2(\theta)$ ... (3)

Substituting (1), (2), and (3) into J, we get

$$J = \int \frac{1}{\cancel{\frac{1}{2}} \tan(\theta) \cdot \left(\sec^2(\theta)\right)^2} \cdot \cancel{\frac{1}{2}} \sec^2(\theta)\, d\theta$$

$$= \int \frac{\sec^2(\theta)}{\tan(\theta) \sec^4(\theta)}\, d\theta = \int \frac{1}{\tan(\theta)\sec^2(\theta)}\, d\theta$$

But $\tan(\theta) = \frac{\operatorname{Sin}(\theta)}{\operatorname{Cos}(\theta)}$, and $\sec(\theta) = \frac{1}{\operatorname{Cos}(\theta)}$, hence,

$$J = \int \frac{1}{\frac{\operatorname{Sin}(\theta)}{\operatorname{Cos}(\theta)} \cdot \frac{1}{\operatorname{Cos}^2(\theta)}}\, d\theta = \int \frac{\operatorname{Cos}^3(\theta)}{\operatorname{Sin}(\theta)}\, d\theta$$

$$\therefore \quad g(\theta) = \frac{\operatorname{Cos}^3(\theta)}{\operatorname{Sin}(\theta)}$$

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8. $$I = \int_1^{\infty} \frac{d}{dx}\left\{\frac{6x + \ln(x)}{2x+1}\right\} dx$$

Integral is Improper of type (I).

$$I = \lim_{R \to +\infty} \int_1^{R} \frac{d}{dx}\left\{\frac{6x + \ln(x)}{2x+1}\right\} dx$$

$$= \lim_{R \to +\infty} \left. \frac{6x + \ln(x)}{2x+1} \right|_{x=1}^{x=R}$$

$$I = \lim_{R \to +\infty} \left[\frac{6R + \ln(R)}{2R+1} - \frac{6(1) + \ln(1)}{2(1)+1} \right] = \lim_{R \to +\infty} \frac{6R + \ln(R)}{2R+1} - 2$$

Now, Consider

$$\lim_{R \to +\infty} \frac{6R + \ln(R)}{2R+1} \quad \left(\text{form } \frac{\infty}{\infty}\right),$$

Hence, by L'Hopital's Rule,

$$\lim_{R \to +\infty} \frac{6R + \ln(R)}{2R+1} = \lim_{R \to +\infty} \frac{6 + \frac{1}{R}}{2}$$

$$= \frac{6+0}{2} = 3$$

It follows that

$$I = 3 - 2 = 1$$

∴ Improper Integral Converges to 1

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9. $I = \int_0^2 \frac{1}{(x-1)^2} \, dx$

Integral is Improper of type (II) since $\frac{1}{(x-1)^2}$ has an infinite discontinuity at $x = 1 \in [0,2]$.

Write $I = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx = I_1 + I_2$

Now, $I_1 = \int_0^1 (x-1)^{-2} dx = \lim_{c \to 1^-} \int_0^c (x-1)^{-2} dx$

$$= -\lim_{c \to 1^-} (x-1)^{-1} \Big|_{x=0}^{x=c}$$

$$I_1 = -\lim_{c \to 1^-} \left[\frac{1}{c-1} + 1 \right] = -(-\infty) = +\infty$$

Similarly $I_2 = +\infty$

Hence, improper integral diverges to $+\infty$.

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10. $J = \int_0^1 (2x^3 - x)\, dx$

Note first that : The integrand is a polynomial in "x" of degree 3. Hence $f^{(4)}(x) \equiv 0$.

Therefore there is no error involved in approximating the integral J using S_{10}. That is to say

$$S_{10} = \text{Exact value of } \int_0^1 (2x^3 - x)\, dx$$

$$= 2\frac{x^4}{4} - \frac{x^2}{2} \Big|_0^1 = \left(\frac{1}{2} - \frac{1}{2}\right) - (0 - 0) = 0$$

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11. Note first that

$$-6 \leq f''(x) \leq 2$$

$$\Rightarrow -6 \leq f''(x) \leq 2 \leq 6$$

$$\Rightarrow -6 \leq f''(x) \leq 6 \quad \text{or} \quad |f''(x)| \leq 6, \; x \in [0, 2]$$

So we take $K = 6$.

Recall $E_T = \frac{K(b-a)^3}{12n^2}$ "Maximum absolute error"

Here $a = 0$, $b = 2$, $K = 6$, and $E_T \leq 0.04$

$$\frac{6(2-0)^3}{12n^2} \le 0.04 \Rightarrow \frac{48}{12n^2} \le \frac{4}{100}$$

$$\Rightarrow \frac{4}{n^2} \le \frac{4}{100} \quad \text{or} \quad \frac{1}{n^2} \le \frac{1}{100}$$

Hence $n^2 \geqslant 100 \Rightarrow n \geqslant 10$.

Therefore the smallest value of n is 10

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12. Let $J = \int_0^1 \int_{-3x}^{x} f(x,y)\,dy\,dx = \iint_R f(x,y)\,dA$,

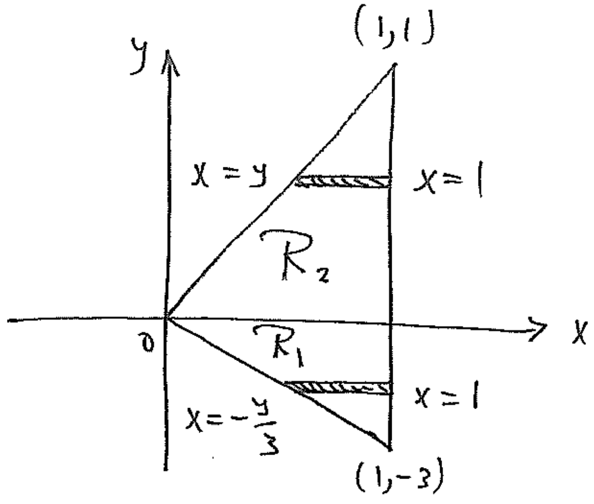
where R is the planar region described by

$-3x \le y \le x, \quad 0 \le x \le 1$ (a y-simple!)

Let us sketch the region R and treat it as an x-simple instead!

$$J = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

$$= \int_{-3}^{0} \int_{-\frac{y}{3}}^{1} f(x,y)\,dx\,dy + \int_{0}^{1} \int_{y}^{1} f(x,y)\,dx\,dy$$



Note: $y = x \Rightarrow x = y$

$y = -3x \Rightarrow x = -\frac{y}{3}$

$R_1 : \begin{cases} -\frac{y}{3} \le x \le 1 \\ -3 \le y \le 0 \end{cases}$

$R_2 : \begin{cases} y \le x \le 1 \\ 0 \le y \le 1 \end{cases}$

$R = R_1 \cup R_2$

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13. Volume $V = \iint\limits_{Base} Height \, dA$

We shall use polar coordinates!

In polar coordinates $x^2+y^2=r^2$, and $dA = r\,dr\,d\theta$

Now, $Z = 3\sqrt{x^2+y^2} \Rightarrow Z = 3r$ --- (1)

and $Z = 6-3(x^2+y^2) \Rightarrow Z = 6-3r^2$ -- (2)

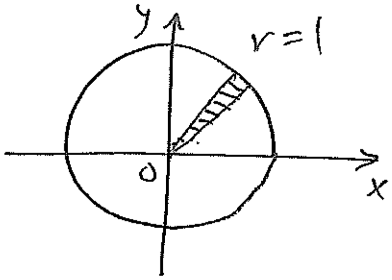
Base: Eliminate "Z" between (1), (2):

$$3r = 6-3r^2 \qquad (\div 3)$$

$$\Rightarrow r^2+r-2=0 \Rightarrow (r+2)(r-1)=0$$

$$r=-2, \quad r=1$$

Reject $r=-2$ since $r>0$. Hence the base is the region enclosed by the circle $r=1$ shown in figure.



$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$

Height: $Z_{top} - Z_{bottom} = (6-3r^2)-(3r)$

$$= 6-3r-3r^2$$

$$\therefore V = \int_0^{2\pi}\int_0^1 (6-3r-3r^2).r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta . \int_0^1 (6r-3r^2-3r^3)\,dr$$

$$= 2\pi\left[3r^2-r^3-\frac{3}{4}r^4\right]_0^1 = 2\pi\left[3-1-\frac{3}{4}\right]$$

$$= 2\pi\left[\frac{5}{4}\right] = \frac{5\pi}{2}$$

14.

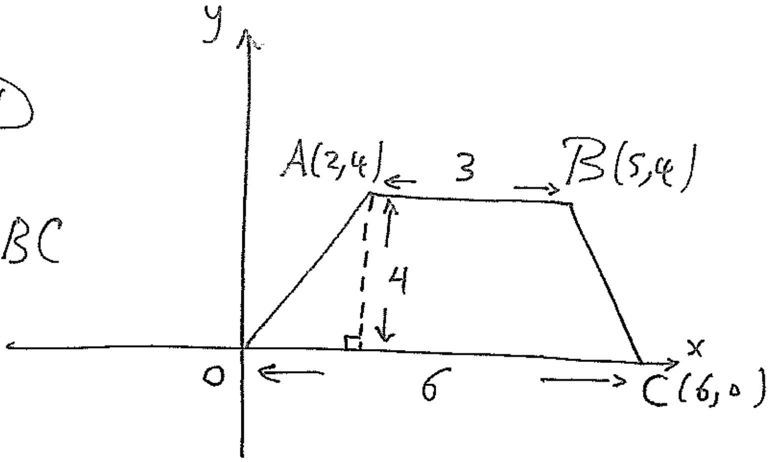
$$\iint_D dA = \text{area "}A\text{" of region } D$$

$$= \text{area of Trapezoid } OABC$$

$$= \tfrac{1}{2}(a+b)\,h$$

$$= \frac{1}{2}(3+6)\cdot 4$$

$$= \frac{1}{2}\cdot 9\cdot 4 = 18$$



a = Minor Base = 3
b = Major Base = 6
h = Height = 4

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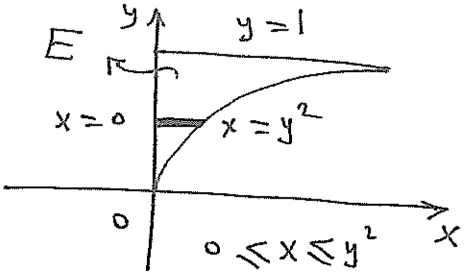
15. For Centroid $\delta(x,y) = 1$.

Hence $dm = \delta(x,y)\,dA = dA$

Mass $m = \iint_E dm = \iint_E dA$

$$= \int_0^1\int_0^{y^2} dx\,dy = \int_0^1 y^2\,dy = \frac{1}{3}y^3\Big|_0^1 = \frac{1}{3},$$



$0 \le x \le y^2$
$0 \le y \le 1$

Next, $M_{Y=0} = \iint_E y\,dm = \iint_E y\,dA = \int_0^1\int_0^{x^2} y\,dx\,dy$

$$= \int_0^1 y\cdot y^2\,dy = \int_0^1 y^3\,dy = \frac{1}{4}y^4\Big|_0^1 = \frac{1}{4}$$

Hence $\bar{y} = \dfrac{M_{Y=0}}{m} = \dfrac{\frac{1}{4}}{\frac{1}{3}} = \dfrac{3}{4}$

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16. $dm = \delta(x,y)\,dA \Rightarrow dm = 8x\,dA$

The region is already described as y- Simple. There is No need to sKetch!

$$\text{Mass } m = \iint_E dm = \iint_E 8x\,dA$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\sin(2x)} 8x\,dy\,dx$$

$$= \int_0^{\frac{\pi}{4}} 8x \cdot y \Big|_{y=0}^{y=\sin(2x)} dx = \int_0^{\frac{\pi}{4}} 8x \sin(2x)\,dx \leftarrow$$

By parts once.

$$\therefore\ m = -4x\cos(2x) + 2\sin(2x) \Big|_{x=0}^{x=\frac{\pi}{4}}$$

$8x$	$\oplus$	$\sin(2x)$
8	$\ominus$	$-\frac{1}{2}\cos(2x)$
0		$-\frac{1}{4}\sin(2x)$

$$= \left[-\pi\cos\left(\tfrac{\pi}{2}\right) + 2\sin\left(\tfrac{\pi}{2}\right)\right] - \left[0 + 2\sin(0)\right]$$

$$= (0+2) - (0+0)$$

$$= 2$$

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University of Calgary - Faculty of Science

Department of Mathematics and Statistics

AMAT 219-L01 - L06 - Winter 2011 - Final Examination

Closed Book , No Calculators.

01. Find $\int \frac{2x}{x^2 - 2x + 10}\, dx$.

02. Let E be the region in the space enclosed by the sphere $(x-1)^2 + y^2 + z^2 = 3$

Find the value of $\iiint_E dV$ is equal to :

03. If $z = \ln(x + y^3)$, find $\frac{\partial^2 z}{\partial y \partial x}$.

04. Let **D** be the planar region occupied by a thin plate with mass equal to 2 units and density function

$\delta(x,y) = y^2$. If $\iint_{\mathbf{D}} xy^2\, dA = 8$ and $\iint_{\mathbf{D}} y^3\, dA = 6$, find the coordinates of the centre of mass of the plate.

05. Find the arc length of the space curve given by the vector equation

$\vec{r}(t) = \frac{1}{3}t^3\, \vec{i} + t^2\, \vec{j} + (2t+1)\, \vec{k}$, $0 \le t \le 3$.

06. A parametric representation of the curve of intersection of the two surfaces

$y + z - x^4 = 3$, and $z = x^2 y + 4$ is given by the vector equation :

(A) $\vec{r}(t) = t\,\vec{i} + (t^2 - 1)\,\vec{j} + (t^4 - t^2 + 4)\,\vec{k}$, $t \in \mathbb{R}$

(B) $\vec{r}(t) = t\,\vec{i} + (t^2 + 1)\,\vec{j} + (t^4 + t^2 + 4)\,\vec{k}$, $t \in \mathbb{R}$, $t \neq \pm 1$

(C) $\vec{r}(t) = t\,\vec{i} + (t^2 - 7)\,\vec{j} + (t^4 - 7t^2 + 4)\,\vec{k}$, $t \in \mathbb{R}$, $t \neq 0$

(D) $\vec{r}(t) = t\,\vec{i} + (7 - t^2)\,\vec{j} + (7t^2 - t^4 + 4)\,\vec{k}$, $t \in \mathbb{R}$, $t \neq 0$

07. Find an equation of the plane tangent to the surface $5x^2 - 2y^2 + 2z = -9$, which is parallel

to the plane $5x - 4y + z = 2$.

08. Let $z = f(x,y) = x^y + y$ where $x = u^2 + v^2 - 4$, and $y = uv - 1$.

Determine the value of $\frac{\partial z}{\partial u}$ at $(u,v) = (2,-1)$.

09. If $W = e^{2x+y}$, where $x = t + \sin(t)$, and $y(t) = 2t - 1$, find the value of $\frac{dW}{dt}$ at $t = 0$.

10. A solid has the shape of the region enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

Find the coordinate of the centroid of the solid.

11. Express the iterated integral $\int_0^1 \int_{z^2}^1 \int_0^{1-y} g(x,y,z)\, dx\, dy\, dz$ as an equivalent integral in which

the $y-$integration is performed first , the $z-$integration second , and the $x-$integration last.

12. The position vector of a moving particle in space is given by the vector equation

$\vec{r}(t) = (t^2 - 1)\ \vec{i} + (8 - 4t)\ \vec{j} + 4\sqrt{3 - t}\ \vec{k}$. Find the speed of the particle at $t = 2$ units.

13. Evaluate the double integral $\int_0^4 \left\{ \int_{\sqrt{y}}^{2} \frac{1}{\sqrt{1 + x^3}}\ dx \right\} dy$ by first reversing the order of the integration.

14. Which of the following plane parametric curves is an equation of an ellipse centred at $(4, -2)$?

(A) $\vec{r}(t) = (3 - 4\cos(t))\ \vec{i} + (5 + 2\sin(t))\ \vec{j}\ ,\quad 0 \le t \le 2\pi.$

(B) $\vec{r}(t) = (3 + 4\cos(t))\ \vec{i} + (5 - 2\sin(t))\ \vec{j}\ ,\quad 0 \le t \le 2\pi.$

(C) $\vec{r}(t) = (4 + 3\cos(t))\ \vec{i} + (-2 + 5\sin(t))\ \vec{j}\ ,\quad 0 \le t \le 2\pi.$

(D) $\vec{r}(t) = (-4 + 3\cos(t))\ \vec{i} + (2 + 5\sin(t))\ \vec{j}\ ,\quad 0 \le t \le 2\pi.$

(E) $\vec{r}(t) = (4 - 2\cos(t))\ \vec{i} + (4 - 2\sin(t))\ \vec{j}\ ,\quad 0 \le t \le 2\pi.$

15. Find $\int \frac{1}{(1 - x^2)^{3/2}}\ dx.$

16. Evaluate $\int (3x^2 + 6x)\ e^x dx.$

17. Find parametric equations of the straight line tangent to the space curve

$\vec{r}(t) = (t^2 + 3t + 1)\ \vec{i} + (2 - 7t)\ \vec{j} + (4\sin(t) - 3)\ \vec{k}$

at the point on the curve corresponding to $t = 0$.

18. Determine whether the improper integral $\int_0^\infty \frac{2x}{(x^2 + 3)^2}\ dx$ converges or diverges and find its value if it exists.

19. Which of the following statements is **True**?

(I) $x^2 - 4y^2 - 9z^2 + 36 = 0$ is an equation of a Hyperboloid of **Two Sheets**.

(II) $x^2 = 2 - y^2 - z^2$ is an equation of a **Sphere**.

(III) $z = \sqrt{1 - x^2 - y^2}$ is an equation of a **Circular Cone**.

(IV) $z = y^2$ is an equation of a **Paraboloid**.

20. A solid **S** has the shape of the region enclosed by the sphere $\rho = \cos(\phi)$.

If the density function $\delta(\rho, \phi, \theta) = 3\cos\left(\frac{\theta}{4}\right)$. Find the mass of the solid **S**.

21. Given that the relation $y^2 + y\sqrt{z} = 14 - \sin(xz^2) + \frac{4}{z}$ implicitly defines x as a differentiable function of y and z. Find the value of $\frac{\partial x}{\partial y}$ at the point $P(0, 3, 4)$.

22. Find the x and y coordinates of all critical point of the function $f(x, y) = 2x^3y - 4x^3 + 6y^3 - 18y + 19$.

Solutions to AMAT 219 Final

1. $\displaystyle\int \frac{2x}{x^2-2x+10}\,dx$

Observe first that $x^2-2x+10$ is not factorable over the real number system. Hence, let us first Complete the square!

Recall $\displaystyle Ax^2+Bx+C=\left(\sqrt{A}\,x+\frac{B}{2\sqrt{A}}\right)^2+\left(C-\frac{B^2}{4A}\right)$, $A>0$

$$=\left(\sqrt{1}\;x+\frac{-2}{2\sqrt{1}}\right)^2+\left(10-\frac{(-2)^2}{4(1)}\right)$$

$$=(x-1)^2+9$$

$$I=\int\frac{2x}{(x-1)^2+9}\,dx.$$

let $t=x-1$, hence $x=t+1$, $dx=dt$

$$\therefore\; I=\int\frac{2(t+1)}{t^2+9}\,dt=\int\frac{2t+2}{t^2+9}\,dt$$

$$=\int\frac{2t}{t^2+9}\,dt+2\int\frac{1}{t^2+3^2}\,dt$$

$$=\ln(t^2+9)+\frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+C$$

Recall $t=x-1$, we have

$$I=\ln\left((x-1)^2+9\right)+\frac{2}{3}\tan^{-1}\left(\frac{x-1}{3}\right)+C$$

or

$$=\ln\left(x^2-2x+10\right)+\frac{2}{3}\tan^{-1}\left(\frac{x-1}{3}\right)+C$$

2. $\iiint_E dV$ = volume V of region E

= Volume of a sphere of radius $\sqrt{3}$

$$= \frac{4\pi}{3}(\sqrt{3})^3 = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}\sqrt{3} = 4\pi\sqrt{3}$$

— — — — — — —

3. $$z = \ln(x+y^3) \Rightarrow \frac{\partial z}{\partial x} = \frac{1}{x+y^3} = (x+y^3)^{-1}$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(x+y^3)^{-1} = -(x+y^3)^{-2}.(3y^2)$$

or $$\frac{\partial^2 z}{\partial y \partial x} = -\frac{3y^2}{(x+y^3)^2}.$$

— — — — —

4. $dm = \delta(x,y)\,dA = y^2\,dA$

Now, $$8 = \iint_D xy^2\,dA = \iint_D x(y^2\,dA) = \iint_D x\,dm = M_{x=0}$$

Likewise:

$$6 = \iint_D y^3\,dA = \iint_D y(y^2\,dA) = \iint_D y\,dm = M_{y=0}$$

$$\therefore \bar{x} = \frac{M_{x=0}}{m} = \frac{8}{2} = 4, \text{ and}$$

$$\bar{y} = \frac{M_{y=0}}{m} = \frac{6}{2} = 3$$

$$\therefore (\bar{x}, \bar{y}) = (4, 3)$$

— — — — — —

5. $\vec{r}(t) = \frac{1}{3}t^3 \vec{i} + t^2 \vec{j} + (2t+1)\vec{k}$, $0 \leqslant t \leqslant 3$

or $\vec{r}(t) = \left(\frac{1}{3}t^3, t^2, 2t+1\right)$

$\vec{v}(t) = (t^2, 2t, 2)$

$\|\vec{v}\| = \sqrt{(t^2)^2 + (2t)^2 + 2^2} = \sqrt{t^4 + 4t^2 + 4}$

$= \sqrt{(t^2+2)^2} = |t^2+2| = t^2+2$ Since $t^2+2>0$.

Arc length $L = \int_{t_0}^{t_1} \|\vec{v}\| dt = \int_0^3 (t^2+2) dt$

$= \frac{1}{3}t^3 + 2t \Big|_{t=0}^{t=3} = \frac{1}{3}(3)^3 + 2(3) = 9+6 = 15$

— — — — — —

6. Note first that: The parametizations (A), (B), (C), and (D) has $x = t$.

Now, $y + z - x^4 = 3$ - - - (1)

$z = x^2 y + 4$ - - - (2)

substituting $x = t$ into (1), (2) we respectively have:

$y + z - t^4 = 3$ - - - (3)

and $z = t^2 y + 4$ - - - (4)

let us solve (3), (4) for y, and z as follows:

Substituting z from (4) into (3), one obtains:

$y + (t^2 y + 4) - t^4 = 3$

$\Rightarrow y + t^2 y = 3 - 4 + t^4$

$y(t^2+1) = t^4 - 1$

$$\therefore\ y = \frac{t^4-1}{t^2+1} = \frac{(t^2-1)\cancel{(t^2+1)}}{\cancel{t^2+1}}$$

$$\Rightarrow\ y = t^2-1 \qquad , \quad t \in \mathbb{R}$$

It follows from (4) that

$$z = t^2(t^2-1)+4 = t^4-t^2+4$$

$\therefore$ A parametrization for the curve is given by

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$$
$$= t\vec{i} + (t^2-1)\vec{j} + (t^4-t^2+4)\vec{k} \ , \quad t \in \mathbb{R}$$

(choice (A)).

---

7. A vector normal to Tangent plane to surface

$$5x^2-2y^2+2z = -9$$

at an arbitrary point is given by

$$\vec{N} = \text{grad}\, F$$

where $F = 5x^2-2y^2+2z+9$

$$\therefore\ \vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (10x, -4y, 2)$$

On the other hand, the Tangent plane is parallel to the plane $5x-4y+z=2$, hence

$$\vec{N} = K(5,-4,1) \qquad \text{for Some } K \in \mathbb{R}$$

That is $(10x, -4y, 2) = K(5,-4,1)$

$$\therefore\ (10x, -4y, 2) = (5K, -4K, K)$$

It follows that

$$10x = 5K \quad \Rightarrow x = \frac{K}{2}$$
$$-4y = -4K \quad \Rightarrow y = K$$

and $\quad 2 = K$

Substituting $K = 2$ into x, and y above, one obtains

$$x = \frac{2}{2} = 1, \text{ and } y = 2$$

To find z: Substitute $x = 1$, $y = 2$ into Equation of surface: $\quad 5x^2 - 2y^2 + 2z = -9$

we get: $\quad 5(1)^2 - 2(2)^2 + 2z = -9$

$$5 - 8 + 2z = -9$$
$$2z = -9 - 5 + 8 = -6 \quad \Rightarrow z = -3$$

$\therefore$ A point on tangent plane is $(x, y, z) = (1, 2, -3)$

The Equation of tangent plane is thus given by

$$5x - 4y + z + d = 0$$

To find d: substitute $(x, y, z) = (1, 2, -3)$:

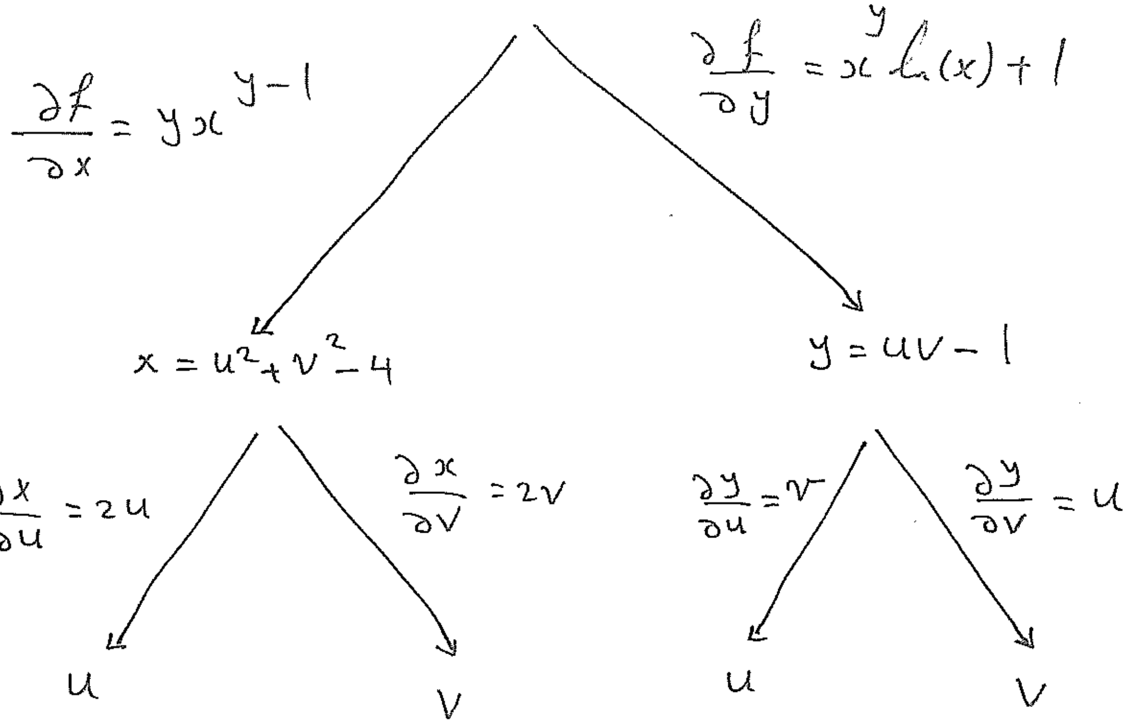
$$5(1) - 4(2) + (-3) + d = 0$$
$$5 - 8 - 3 + d = 0 \Rightarrow d = 6$$

$\therefore$ Equation of tangent plane is

$$5x - 4y + z + 6 = 0$$

8.

$$Z = f(x,y) = x^{y} + y$$



$$\frac{\partial z}{\partial u} = y x^{y-1} \cdot 2u + \left(x^{y} \ln(x) + 1\right) \cdot v \Bigg|_{\substack{u=2\\ v=-1\\ x=1\\ y=-3}}$$

$$= (-3)(1)\,2(2) + \left(1 \cdot \ln(1) + 1\right)(-1)$$

$$= -12 - 1 = -13$$

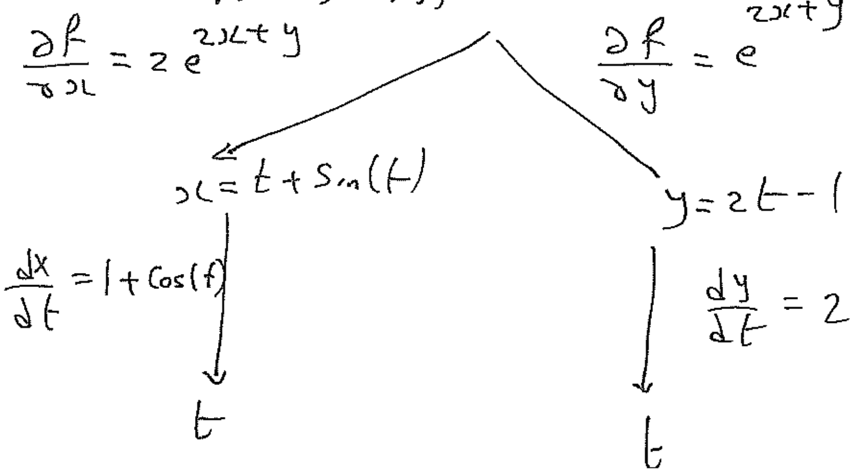
Note:
$x = u^2 + v^2 - 4$
At $u = 2$, $v = -1$,
$x = 2^2 + (-1)^2 - 4 = 1$

$y = uv - 1$
At $u = 2$, $v = -1$
$y = 2(-1) - 1 = -3$

- - - - - - - - - -

9.

$$W = f(x,y) = e^{2x+y}$$



Note
$x = t + \sin(t)$
At $t = 0$,
$x = 0 + \sin(0) = 0$

$y = 2t - 1$
At $t = 0$,
$y = 2(0) - 1 = -1$

$$\therefore \quad \frac{dW}{dt} = 2e^{2x+y}\cdot(1+\cos(t)) + e^{2x+y}\cdot 2 \Bigg|_{\substack{t=0\\ x=0\\ y=-1}}$$

$$= 2e^{0+(-1)}(1+\cos(0)) + e^{0+(-1)}\cdot 2$$

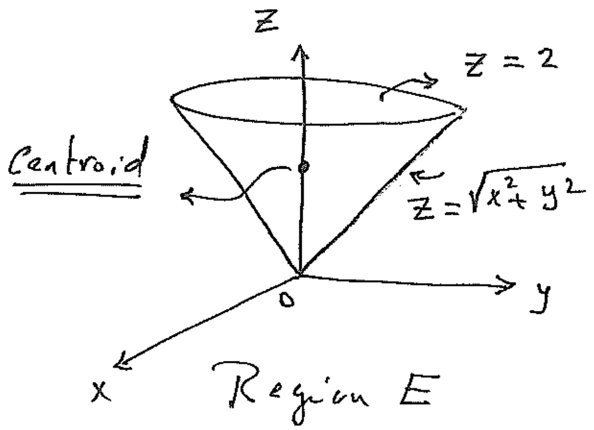
$$= 2e^{-1}(1+1) + e^{-1}\cdot 2$$

$$= 4e^{-1} + 2e^{-1} = 6e^{-1} \overset{\text{or}}{=} \frac{6}{e}$$

— — — — — — —

10. First observe that from symmetry, the centroid of the solid lies on the z-axis as evident from figure below. Therefore $\bar{x} = 0$, $\bar{y} = 0$. We need only to compute

$$\bar{z} = \frac{M_{z=0}}{m}.$$



For centroid $\delta(x,y,z) = 1$, hence

$$dm = \delta(x,y,z)\,dV = dV$$

$$\text{Mass} \quad m = \iiint_E dm$$

$$\therefore \quad m = \iiint_E dV \quad \text{---} \quad (1)$$

$$M_{z=0} = \iiint_E z\,dm$$

$$\therefore \quad M_{z=0} = \iiint_E z\,dV \quad \text{--} \quad (2)$$

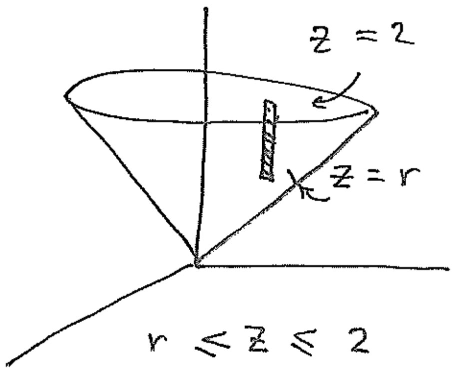
We shall use cylindrical Coordinates to Compute triple Integrals (1), (2).

In cylindrical coordinates:

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$,

$x^2 + y^2 = r^2$, and $dV = dz\,dA$, where $dA = r\,dr\,d\theta$

Now, $z = \sqrt{x^2+y^2} \Rightarrow z = \sqrt{r^2} = r$

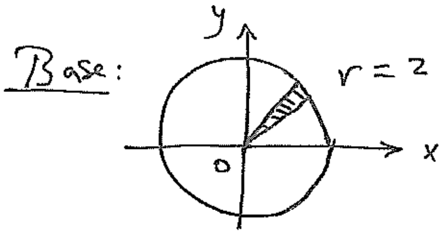


$r \leq z \leq 2$

Base: $z = r$, $z = 2$.

Eliminating "z", we have

$r = 2$. So base is the region enclosed by the circle Centred at origin and is of radius 2



$0 \leq r \leq 2$
$0 \leq \theta \leq 2\pi$

Now,

$$m = \iiint_E dV$$

$$= \iint_{Base} \left\{ \int_r^2 dz \right\} dA = \iint_{Base} (2-r)\,dA$$

$$= \int_0^{2\pi} \int_0^2 (2-r).r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 (2r - r^2)\,dr = 2\pi \left[r^2 - \frac{1}{3} r^3 \right]_0^2$$

$$= 2\pi \left[2^2 - \frac{1}{3} \cdot 2^3 \right] = 2\pi \left[4 - \frac{8}{3} \right]$$

$$= 8\pi \left[1 - \frac{2}{3} \right] = 8\pi \cdot \frac{1}{3} = \frac{8\pi}{3}$$

<u>Note</u>: We could have computed m as follows:

$$m = \iiint_E dV = \text{volume of region } E$$

$= \text{volume of a cone of radius } r = 2,$ and height $h = 2$

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (2)^2 \cdot 2 = \frac{8\pi}{3}$$

Next, $M_{z=0} = \iiint_E z\,dV = \iint_{Base} \left\{ \int_r^2 z\,dz \right\} dA$

$$= \iint_{Base} \frac{1}{2} z^2 \Big|_r^2 \, dA = \frac{1}{2} \iint_{Base} (2^2 - r^2)\, dA$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta \cdot \int_0^2 (4r - r^3)\, dr$$

$$= \frac{1}{2} \cdot 2\pi \left[2r^2 - \frac{1}{4} r^4 \right]_0^2$$

$$= \pi \left[2(2)^2 - \frac{1}{4}(2)^4 \right] = \pi \left[8 - 4 \right] = 4\pi$$

$$\therefore \quad \bar{z} = \frac{M_{z=0}}{m} = \frac{4\cancel{\pi}}{\frac{8\cancel{\pi}}{3}} = \frac{(4)(3)}{8} = \frac{3}{2}$$

Centroid is at the point $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{3}{2})$

---

11. Let $I = \int_0^1 \int_{z^2}^1 \int_0^{1-y} g(x,y,z)\, dx\, dy\, dz$

$$= \iiint_E g(x,y,z)\, dV$$

where E is the region described by

$0 \leq x \leq 1-y$ --- (1)

$z^2 \leq y \leq 1$ --- (2)

$0 \leq z \leq 1$ --- (3)

In the new order : $dV = dy\, dz\, dx$

Now, from (2) : $z^2 \leq y$

and from (1) : $x \leq 1-y \Rightarrow y \leq 1-x$

It follows that

$$\boxed{z^2 \leq y \leq 1-x} \quad \text{--- } (*)$$

<u>Next</u> : From (3) : $0 \leq z$

and from (*) $z^2 \leq 1-x \Rightarrow z \leq \sqrt{1-x}$

It follows that

$$\boxed{0 \leq z \leq \sqrt{1-x}} \quad \text{--- } (**)$$

Finally, from (1) : $0 \leq x \leq 1-y \leq 1$ " Since $y \geq 0$ "

$$\therefore \boxed{0 \leq x \leq 1} \quad (***)$$

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x}} \int_{z^2}^{1-x} g(x,y,z)\, dy\, dz\, dx$$

12. $\vec{r}(t) = (t^2-1)\vec{i} + (8-4t)\vec{j} + 4\sqrt{3-t}\,\vec{k}$

$$\overset{\text{or}}{=} \left(t^2-1,\ 8-4t,\ 4(3-t)^{\frac{1}{2}} \right)$$

Velocity $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \left(2t, -4, 4\cdot\frac{1}{2}(3-t)^{-\frac{1}{2}}(-1) \right)$

$$= \left(2t, -4, -\frac{2}{\sqrt{3-t}} \right)$$

At $t=2$, $\vec{v}(2) = \left(2(2), -4, -\dfrac{2}{\sqrt{3-2}} \right)$

$$= (4, -4, -2)$$

speed $= \|\vec{v}(2)\| = \sqrt{4^2+(-4)^2+(-2)^2}$

$$= \sqrt{16+16+4} = \sqrt{36} = 6$$

13. $I = \displaystyle\int_0^4 \left\{ \int_{\sqrt{y}}^{2} \frac{1}{\sqrt{1+x^3}}\,dx \right\} dy = \iint_R \frac{1}{\sqrt{1+x^3}}\,dA,$

where R is the x-Simple region described by

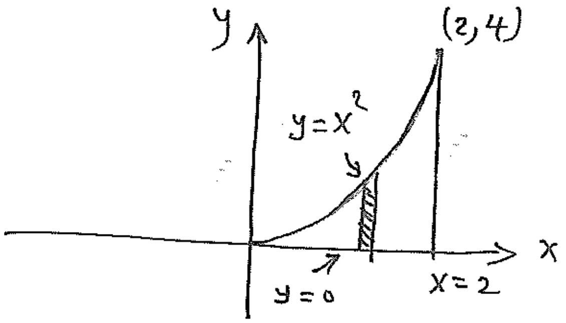
$$\sqrt{y} \leq x \leq 2, \quad 0 \leq y \leq 4$$

let us sketch region R and Treat it as y-Simple instead!

<u>Note</u>: $x = \sqrt{y} \Rightarrow x^2 = y$

$$\therefore I = \int_0^2 \left\{ \int_0^{x^2} \frac{1}{\sqrt{1+x^3}}\,dy \right\} dx$$

$$= \int_0^2 \frac{1}{\sqrt{1+x^3}} \left\{ \int_0^{x^2} dy \right\} dx$$



$0 \leq y \leq x^2$
$0 \leq x \leq 2$

$$I = \int_0^2 \frac{1}{\sqrt{1+x^3}} \cdot x^2\,dx$$

let $t = 1+x^3$, $\therefore\ dt = 3x^2\,dx \Rightarrow x^2\,dx = \frac{1}{3}\,dt$

$$\therefore\ I = \frac{1}{3}\int_1^9 \frac{1}{\sqrt{t}}\,dt = \frac{1}{3}\int_1^9 t^{-\frac{1}{2}}\,dt$$

$$= \frac{2}{3}\sqrt{t}\,\Big|_1^9 = \frac{2}{3}\left[\sqrt{9}-\sqrt{1}\right]$$

$$= \frac{2}{3}[3-1] = \frac{4}{3}$$

New limits

$t = 1+x^3$

at $x = 0$,

$t = 1+0^3 = 1$

at $x = 2$,

$t = 1+2^3 = 9$

— — — — — — —

14. Recall: The vector equation of an ellipse Centred at $(h,K) = (4,-2)$ and semi-axes of length a, b is given by

$$\vec{r}(t) = \left(4 + a\cos(t)\right)\vec{i} + \left(-2+b\sin(t)\right)\vec{j}, \quad t\in[0,2\pi]$$

In particular if $a = 3$, $b = 5$, we have

$$\vec{r}(t) = \left(4 + 3\cos(t)\right)\vec{i} + \left(-2+5\sin(t)\right)\vec{j}, \quad t\in[0,2\pi]$$

So: Answer is (C).

— — — — — — —

15. $I = \int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$

The integrand contains the special quantity

$$a^2 - b^2x^2 = 1 - x^2 \quad (\text{with } a = 1,\ b = 1)$$

$\therefore$ We let, $x = \frac{a}{b}\sin(\theta)$, that is

$$x = \sin(\theta) \quad ---(1)$$

$$\therefore \quad dx = \cos(\theta)\, d\theta \quad ---(2)$$

and $$1 - x^2 = 1 - \sin^2(\theta)$$

$$\therefore \quad 1 - x^2 = \cos^2(\theta) \quad ---(3)$$

Substituting (2), (3) into I, we get

$$I = \int \frac{1}{(\cos^2(\theta))^{\frac{3}{2}}} \cdot \cos(\theta)\, d\theta$$

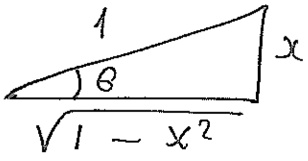
$$= \int \frac{\cos(\theta)}{\cos^3(\theta)}\, d\theta = \int \frac{1}{\cos^2(\theta)}\, d\theta$$

$$= \int \sec^2(\theta)\, d\theta$$

$$= \tan(\theta) + C$$

$$\overset{\text{From}}{\underset{\Delta}{=}} \frac{x}{\sqrt{1-x^2}} + C$$

$$\sin(\theta) = x = \frac{x}{1}$$



$$\tan(\theta) = \frac{x}{\sqrt{1-x^2}}$$

16. $\int (3x^2 + 6x)e^x\, dx$ ← By parts

$$I = (3x^2+6x)e^x - (6x+6)e^x + 6e^x + C$$

$$= e^x\left[3x^2 + \cancel{6x} - \cancel{6x} - \cancel{6} + \cancel{6}\right] + C$$

$$= 3x^2 e^x + C$$

$$\begin{array}{ccc} 3x^2+6x & \oplus & e^x \\ 6x+6 & \ominus & e^x \\ 6 & \oplus & e^x \\ 0 & & e^x \end{array}$$

17. $\vec{r}(t) = \left(t^2+3t+1,\ 2-7t,\ 4\sin(t)-3 \right)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(2t+3,\ -7,\ 4\cos(t) \right)$$

At $t=0$, $\vec{r}(0) = \left(0+0+1,\ 2-0,\ 4\sin(0)-3 \right)$

$$= (1, 2, -3)$$

and $\vec{v}(0) = \left(0+3,\ -7,\ 4\cos(0) \right)$

$$= (3, -7, 4)$$

Vector equation of line tangent to curve is thus given by $\vec{r} = \vec{r}(0) + s\,\vec{v}(0)$, $s \in \mathbb{R}$

$$\therefore\ (x, y, z) = (1, 2, -3) + s(3, -7, 4)$$

$$\therefore\ \begin{aligned} x &= 1+3s \\ y &= 2-7s \\ z &= -3+4s \end{aligned}, \quad s \in \mathbb{R}$$

---

18. $I = \displaystyle\int_0^\infty \frac{2x}{(x^2+3)^2}\,dx$

Integral is Improper of type (I).

Now, $I = \displaystyle\lim_{R\to+\infty} \int_0^R \frac{2x}{(x^2+3)^2}\,dx$ --- $(*)$

Consider $J = \displaystyle\int_0^R \frac{2x}{(x^2+3)^2}\,dx$ ← Substitution

Let $u = x^2+3$

$\therefore\ du = 2x\,dx$

$$\therefore\ J = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{x^2+3}\Bigg|^{x=R} = -\left[\frac{1}{R^2+3} - \frac{1}{3}\right]$$

substituting into (*),

$$I = \lim_{R\to+\infty}\left[\frac{-1}{R^2+3}+\frac{1}{3}\right] = 0+\frac{1}{3} = \frac{1}{3}$$

Hence Improper Integral Converges to $\frac{1}{3}$.

— — — — — — —

19. (I) $x^2-4y^2-9z^2+36=0$

$$\Rightarrow -x^2+4y^2+9z^2=36$$

which is an equation of a Hyperboloid of <u>one</u> sheet.

Statement is False!

(II) $x^2=2-y^2-z^2$

$$\Rightarrow x^2+y^2+z^2=2$$

which is an equation of a sphere

Statement is True!

(III) $z=\sqrt{1-x^2-y^2}$

$$\Rightarrow z^2=1-x^2-y^2$$

$$\Rightarrow x^2+y^2+z^2=1$$ .. An Equation of a sphere

Therefore $z=\sqrt{1-x^2-y^2}$ is an Equation of an upper hemi-sphere.

Statement is False!

(IV) $z=y^2$ is an Equation of a cylinder!

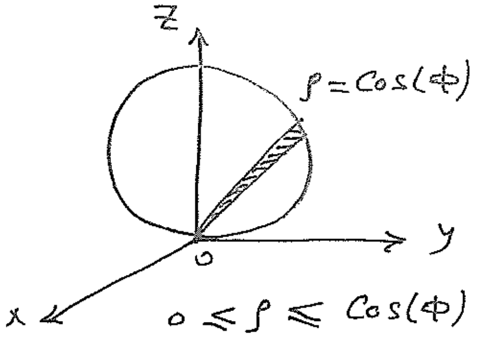
Statement is False!

So: Only Statement (II) is True!

20. In spherical Coordinates

$$dm = \delta(\rho, \phi, \theta)\, dV$$

$$= 3\cos\left(\frac{\theta}{4}\right) \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$



$0 \leqslant \rho \leqslant \cos(\phi)$
$0 \leqslant \phi \leqslant \frac{\pi}{2}$
$0 \leqslant \theta \leqslant 2\pi$

Therefore,

$$\text{Mass } m = \iiint_S dm$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\cos(\phi)} 3\cos\left(\frac{\theta}{4}\right) \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3\cos\left(\frac{\theta}{4}\right) \sin(\phi) \left\{ \int_0^{\cos(\phi)} \rho^2 d\rho \right\} d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cancel{3} \cos\left(\frac{\theta}{4}\right) \sin(\phi) \cdot \frac{1}{\cancel{3}} \rho^3 \Big|_0^{\cos(\phi)} d\phi\, d\theta$$

$$= \int_0^{2\pi} \cos\left(\frac{\theta}{4}\right) d\theta \cdot \int_0^{\frac{\pi}{2}} \cos^3(\phi) \sin(\phi)\, d\phi$$

$$= 4\sin\left(\frac{\theta}{4}\right) \Big|_0^{2\pi} \cdot J = 4\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] J$$

$$= 4[1-0] J = 4J,$$

where $J = \int_0^{\frac{\pi}{2}} \cos^3(\phi) \sin(\phi)\, d\phi$

let $t = \cos(\phi)$, $\therefore\ dt = -\sin(\phi) d\phi$

or $\sin(\phi) d\phi = -dt$

$$\therefore J = -\int t^3\,dt = -\frac{1}{4}t^4 = -\frac{1}{4}\cos^4(\phi)\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{4}\left[\cos^4\left(\frac{\pi}{2}\right) - \cos^4(0)\right]$$

$$= -\frac{1}{4}[0 - 1] = \frac{1}{4}$$

Hence $m = 4J = 4 \cdot \frac{1}{4} = 1$

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21. $y^2 + y\sqrt{z} = 14 - \sin(xz^2) + \frac{4}{z}$

$$\Rightarrow y^2 + y\sqrt{z} + \sin(xz^2) - \frac{4}{z} - 14 = 0$$

let $F(x,y,z) = y^2 + y\sqrt{z} + \sin(xz^2) - \frac{4}{z} - 14$

Therefore

$$\frac{\partial x}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} \quad , \text{ provided } \frac{\partial F}{\partial x} \neq 0$$

$$= -\frac{2y + \sqrt{z} + 0 - 0 - 0}{0 + 0 + z^2\cos(xz^2) - 0 - 0}$$

$$= -\frac{2y + \sqrt{z}}{z^2\cos(xz^2)}$$

At $P(0,3,4)$, we have

$$\frac{\partial x}{\partial y} = -\frac{2y+\sqrt{z}}{z^2\cos(xz^2)}\Bigg|_{\substack{x=0\\y=3\\z=4}} = -\frac{2(3)+\sqrt{4}}{4^2\cos(0)} = -\frac{8}{16}$$

$$= -\frac{1}{2}$$

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22. $f(x,y) = 2x^3y - 4x^3 + 6y^3 - 18y + 19$

$$\frac{\partial f}{\partial x} = 6x^2y - 12x^2,$$

$$\frac{\partial f}{\partial y} = 2x^3 + 18y^2 - 18$$

Critical points occur where

$$\frac{\partial f}{\partial x} = 0, \text{ and } \frac{\partial f}{\partial x} = 0$$

That is $6x^2y - 12x^2 = 0 \; - - - (1)$

$2x^3 + 18y^2 - 18 = 0 \; - - - (2)$

let us solve system (1), (2) above!

From (1): $6x^2y - 12x^2 = 0$

$\Rightarrow 6x^2(y-2) = 0$

∴ Either $\underline{x = 0}$, or $\underline{y = 2}$

Case 1: If $x = 0$, Equation (2) reduces to

$0 + 18y^2 - 18 = 0 \Rightarrow y^2 - 1 = 0$

$\Rightarrow y = \pm 1$

Therefore: Critical points are $(0, 1)$, $(0, -1)$.

Case 2: If $y = 2$, equation (2) reduces to

$2x^3 + 18(2)^2 - 18 = 0 \Rightarrow 2x^3 + 54 = 0$

$\Rightarrow x^3 = -27 \Rightarrow x = -3$

∴ $(x,y) = (-3, 2)$ is a critical point.

Conclusion: There are three critical points:

$(0, 1)$, $(0, -1)$, and $(-3, 2)$