MAT 1341 Midterm Test #5 Solutions, Version 1

November 17, 2014

(1) **[1 point]** For which value(s) of a are the vectors $\{(a, 1, 0), (1, 2, 3), (3, 2, 1)\}$ linearly dependent?

A. $a = 2$
B. $a = 0$
C. $a \neq 0$
D. $a = -2$
E. For all values of a.
F. For no value of a.

Solution: A. Let A be the matrix whose columns are the three given vectors. Calculate $det(A) = a(2 \cdot 1 - 2 \cdot 3) - 1(1 \cdot 1 - 3 \cdot 3) = -4a + 8$. The vectors are dependent precisely when A is not invertible, i.e. when $det(A) = 0$. That happens precisely when $a = 2$.

(2) **[1 point]** Which of the following is **not** a subspace of $\mathbb{R}^3$?

A. $\{(x, y, z)| -x + 2y = 0\}$
B. $\{(x, y, z)|x = 0 \text{ and } y + z = 0\}$
C. $\{(x, y, z)|x - y = z\}$
D. $\{(x, y, z)|x \leq y\}$
E. $\{(x, y, z)|(x, y, z) \cdot (1, 2, 3) = 0\}$
F. $\{(x, y, z)|x + y + z = 0 \text{ and } x - z = 0\}$.

Solution: A,B,C, and F are given as sets of solutions to homogeneous systems and hence are subspaces. E can be rewritten as $\{(x, y, z)|x + 2y + 3z = 0\}$, so that's a subspace as well. That leaves D: this is not a subspace because it isn't closed under scalar multiplication. E.g., $(1, 2, 3) \in D$, but $-(1, 2, 3) = (-1, -2, -3)$ is not in D.

(3) **[1 point]** Exactly one of the following statements is true. Which one?

A. The set $\{x \in \mathbb{R}^n|Ax = b\}$ of solutions to a linear system is always a subspace of $\mathbb{R}^n$.

B. For any set $\{x_1, \ldots, x_n\}$ of vectors in $\mathbb{R}^n$, $Span\{x_1, \ldots, x_n\} = \mathbb{R}^n$.

C. If $\{v_1, v_2, v_3\}$ is linearly dependent, then v_3 is a linear combination of v_1 and v_2.

D. If an nxn matrix has rank $< n$, then the columns are linearly dependent.

E. If A is an nxn matrix, then the columns of A form a subspace of $\mathbb{R}^n$.

F. If $det(A) = 0$, then the null space $null(A)$ of A is empty.

Solution: D is true: if the rank is less than n, one of the columns has no leading 1, and that column is a linear combination of the other columns.

(4) **[1 point]** Suppose $\{v_1, v_2\}$ is a linearly independent set of vectors in $\mathbb{R}^3$. Which of the following sets of vectors is then also linearly independent?

A. $\{v_1, v_2, v_1 + v_2\}$

B. $\{v_1, v_2, v_1 - v_2\}$

C. $\{v_1 - v_2, v_2 - v_1\}$

D. $\{v_1 - v_2, v_2 + v_1\}$

E. $\{v_1, v_1 + v_2, v_1 - v_2\}$

F. None of the above.

Solution: D is: if $a(v_1 - v_2) + b(v_2 + v_1) = 0$, then $(a + b)v_1 q + (b - a)v_2 = 0$. That implies $a + b = 0, b - a = 0$, which in turn gives $a = b = 0$.

(5) **[1 point]** Which of the following is a basis for the subspace $\{(x, y, z, w)|3x - 2y + z = 5w\}$ of $\mathbb{R}^4$?

A. $\{(3, -2, 1, -5)\}$

B. $\{1, -1, 0, 1)\}$

C. $\{3, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0)\}$

D. $\{3, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0), (0, 0, 0, -5)\}$

E. $\{2, 3, 0, 0), (1, 0, -3, 0), (5, 0, 0, 3)\}$

F. None of the above.

Solution: The subspace has dimension 3 (because there is one leading 1 and 3 columns without leading 1), and thus a basis will have 3 elements. These have to be vectors in the subspace, so have to satisfy the given equation. That leaves E as the only possible answer.

(6) Consider the matrix $M = \begin{bmatrix} 2 & 4 & 8 & -2 \\ -2 & 1 & 2 & 2 \\ 1 & 3 & 6 & -1 \end{bmatrix}$.

(a) **[4 points]** Find a basis for the column space $Col(A)$ of A.

Solution: Row-reduce M:

$$\begin{bmatrix} 2 & 4 & 8 & -2 \\ -2 & 1 & 2 & 2 \\ 1 & 3 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 5 & 10 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two columns have a leading 1, so a basis for $Col(A)$ is $\{(2, -2, 1), (4, 1, 3)\}$.

Marking: 2 points for row-reducing, one for understanding that you must look at the columns with leading 1s, and 1 for the correct conclusion.

(b) **[2 points]** Give a complete geometric description of $Col(A)$.

Solution: $Col(A)$ is a plane through the origin with direction vectors $(2, -2, 1)$ and $(4, 1, 3)$.

Marking: 1 point for recognizing it is a plane, 1 point for saying which one exactly.

(7) Let A be a 3x3 matrix, and consider the set $V = \{x \in \mathbb{R}^n | Ax = x\}$.

(a) [**4 points**] Prove **using the definition of subspace** that V is a subspace of $\mathbb{R}^n$.

Solution:
(i) Taking $x = 0$, we get $A0 = 0$, so $0 \in V$.
(ii) If $x, y \in V$, then we have $Ax = x, Ay = y$. Then $A(x + y) = Ax + Ay = x + y$, which shows that $x + y \in V$.
(iii) If $x \in V$ and $c \in \mathbb{R}$, then $Ax = x$, so $A(cx) = cAx = cx$, whence $cx \in V$.

Marking: 1 point for knowing the conditions; 1 point for showing how the proof should be structured; 2 points for execution.

(b) [**2 points**] Suppose that $A = I_3$. Give a basis for V.

Solution: In this case, every $x \in \mathbb{R}^3$ satisfies $Ax = x$. Thus $V = \mathbb{R}^3$, and any basis for $\mathbb{R}^3$ will do, e.g. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Marking: 1 point for understanding that $V = \mathbb{R}^3$, 1 point for a basis.

(8) For each of the following statements, say whether it is (always) true or (possibly) false. If true, explain in a few sentences why (referring to facts learned in class or from the book as needed). If false, give a counterexample (with numbers!).

(a) [**1 point**] Any plane in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Solution: False; only planes that pass through the origin are subspaces. For example, the plane $x = 1$ is not a subspace because it doesn't contain $(0, 0, 0)$.

(b) [**1 point**] If $z \in Span\{x, y\}$ then $z \in Span\{x, y + x\}$.

Solution: True; if $z = ax + by$, then we can also write z as a linear combination of $x, y + x$, namely $z = (a - b)x + b(y + x)$.

(c) [**1 point**] The set $\{t(1, 0, 3) | t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Solution: True; this is simply a line through the origin; equivalently, it is $Span\{(1, 0, 3)\}$, and spans are always subspaces.

(d) [**1 point**] If $Span\{x\} = Span\{x, y\}$, then $\{x, y\}$ is linearly dependent.

Solution: True; we have $y \in Span\{x, y\} = Span\{x\}$, so y is a scalar multiple of x.