MAT 1341 Midterm Test #4 Solutions, Version 1

November 3, 2014

(1) Let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $N = \begin{bmatrix} -3a & -3b & -3c \\ d-2g & e-2h & f-2i \\ 2a+g & 2b+h & 2c+i \end{bmatrix}$.

If $det(M) = 5$, then what is $det(N)$?

A. 5
B. -15
C. 15
D. -30
E. 30
F. None of the above.

Solution: B To get N from M, we first subtract two times row 3 from row 2, then add two times row 1 to row 3, and then multiply the first row by -3. Only the second step changes the determinant (by a factor -3), so $det(N) = -3det(M) = -15$.

(2) **[1 point]** Which of the following is an eigenvector of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$?

A. $[1,1,1]^T$
B. $[0,0,1]^T$
C. $[0,1,-1]^T$
D. $[1,0,-1]^T$
E. $[1,-1,1]^T$
F. None of the above.

Solution: C Simply test each of the vectors by multiplying with the given matrix on the left. Then you see that when you apply the matrix to $[0,1,-1]$ you get $[0,1,-1]$, so that it is an eigenvector with eigenvalue $\lambda = 1$.

(3) **[1 point]** Exactly one of the following statements is **not** true. Which one?

A. A square matrix A is invertible precisely when $det(A) \neq 0$.

B. Every non-zero vector $\vec{x} \in \mathbb{R}^n$ is an eigenvector of the identity matrix I_n.

C. If $\vec{x}$ is an eigenvector of A and of B, then it is also an eigenvector of AB.

D. If a square matrix has no real eigenvalues, then it cannot be invertible.

E. If A is a 4x4 matrix with $det(A) = 5$, then $det(-A) = 5$.

F. $det(A) = det(A^T)$ for any square matrix A.

Solution: D. For example, a rotation matrix is always invertible but doesn't have any real eigenvalues (unless it's rotation by a multiple of π).

(4) **[1 point]** Consider a matrix A satisfying

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Compute A^{100}.

A. $A^{100} = \begin{bmatrix} -1 & 0 \\ -1 - 2^{100} & 2^{100} \end{bmatrix}$ B. $A^{100} = \begin{bmatrix} 1 & 0 \\ 1 - 2^{100} & 2^{100} \end{bmatrix}$

C. $A^{100} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ D. $A^{100} = \begin{bmatrix} -1 & 0 \\ 3 & 2^{100} \end{bmatrix}$

E. $A^{100} = \begin{bmatrix} 1 & 0 \\ 1 + 2^{100} & 2^{100} \end{bmatrix}$ F. None of the above.

Solution: B Use the fact that if $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$. Then

$$A^{100} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 - 2^{100} & 2^{100} \end{bmatrix}$$

(5) [**2 points**] Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Which of the following statements are true? Circle ALL that are true.

A. The eigenvalues of A are $\lambda = 1, \lambda = 0$.
B. The only eigenvalue of A is $\lambda = 1$.
C. The matrix A has no real eigenvalues.
D. The eigenvalues of B are $\lambda = -1, \lambda = 1$
E. The only eigenvalue of B is $\lambda = -1$.
F. The matrix B has no real eigenvalues.
G. The eigenvalues of C are $\lambda = 0, \lambda = 1$
H. The only eigenvalue of C is $\lambda = \frac{1}{2}$.
I. The matrix C has no real eigenvalues.

Solution: A,F,G All of these can be done by geometric inspection. For A, we see that $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}$. Thus this matrix acts by taking a vector and moving the tip vertically to the line $y = x$. Hence all vectors on $y = x$ remain fixed (so that $\lambda = 1$ is an eigenvalue) and all vectors on $x = 0$ are sent to the origin (so that $\lambda = 0$ is an eigenvalue).

For B, this is rotation by $\pi/2$, so there are no real eigenvalues.

For C, this is orthogonal projection on $y = x$, so the eigenvalues are $0, 1$.

(6) Consider the matrix $M = \begin{bmatrix} 2 & 4 & 8 \\ -2 & 1 & 2 \\ 3 & 3 & -3 \end{bmatrix}$.

(a) **[2 points]** Compute the determinant of M using cofactor expansion.

Solution: Expanding along the first row gives:

$$\begin{vmatrix} 2 & 4 & 8 \\ -2 & 1 & 2 \\ 3 & 3 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} - 4\begin{vmatrix} -2 & 2 \\ 3 & -3 \end{vmatrix} + 8\begin{vmatrix} -2 & 1 \\ 3 & 3 \end{vmatrix} =$$

$$2[1\cdot(-3) - 2\cdot 3] - 4[(-2)\cdot -3 - 2\cdot 3] + 8[(-2)\cdot 3 - 1\cdot 3] = -18 - 0 - 72 = -90$$

Marking: One point for showing the correct method, one for execution. (Subtract 1/2 point for errors with minus signs.)

(b) **[4 points]** Compute the determinant of M by bringing M into row echelon form. Explain clearly what row reduction steps you take and how they change the determinant.

Solution:

$$\begin{vmatrix} 2 & 4 & 8 \\ -2 & 1 & 2 \\ 3 & 3 & -3 \end{vmatrix} \begin{matrix} R_1 := \frac{1}{2}R_1 \\ R_3 := \frac{1}{3}R_3 \\ \sim \end{matrix} \begin{vmatrix} 1 & 2 & 4 \\ -2 & 1 & 2 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} R_2 := R_2 + 2R_1 \\ R_3 := R_3 - R_1 \\ \sim \end{matrix} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 10 \\ 0 & -1 & -5 \end{vmatrix}$$

$$\begin{matrix} R_2 := \frac{1}{5}R_2 \\ \sim \end{matrix} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & -1 & -5 \end{vmatrix} \begin{matrix} R_3 := R_3 + R_2 \\ \sim \end{matrix} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{vmatrix} \begin{matrix} R_3 := -\frac{1}{3}R_3 \\ \sim \end{matrix} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

Since the determinant of the last matrix is 1, we can trace back the steps to get $det(M) = 1\cdot(-3)\cdot 5\cdot 2\cdot 3 = -90$. (The only steps in the sequence which change the determinant are those where we multiply a row by a scalar.)

Marking: 2 points for a clear reduction, 2 for correctly deducing the determinant.

(7) **[3 points]** Suppose A and B are 3x3 matrices with $det(A) = -1$ and $det(B) = 3$. Compute the determinant of $(2A^T)B^{-1}A^3(3B)^2$.

Solution: Use $|A^T| = |A|$, $|B^{-1}| = \frac{1}{|B|}$, $|A^3| = |A|^3$, $|(3B)^2| = |9B^2| = 9^3|B|^2$ to get

$$|(2A^T)B^{-1}A^3(3B)^2| = |(2A^T)|\cdot|B^{-1}|\cdot|A^3|\cdot|(3B)^2| = 2^3|A|\cdot\frac{1}{|B|}\cdot|A|^3\cdot 9^3|B|^2$$

$$= 2^3\cdot(-1)\cdot\frac{1}{3}\cdot(-1)^3\cdot 9^3\cdot 3^2 = 8\cdot 3^7.$$

Marking: 2 points for understanding the rules for the determinant, one for correct calculation.

(8) For each of the following statements, say whether it is (always) true or (possibly) false. If true, explain in a few sentences why (referring to facts learned in class or from the book as needed). If false, give a counterexample (with numbers!).

Marking: 1/2 point for the correct answer, 1/2 point for a valid argumentation.

(a) **[1 point]** If A and B are 3x3 matrices with $det(AB) \neq 0$, then A and B are invertible.

Solution: True: $det(AB) = det(A)det(B)$, so if this product is non-zero, both $det(A) \neq 0$ and $\det(B) \neq 0$, so that A and B are invertible.

(b) **[1 point]** If A and B are 3x3 matrices then $det(A + B) = det(A) + det(B)$.

Solution: False, let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $|A| = |B| = 0$, but $|A + B| = |I_3| = 1$.

(c) **[1 point]** If $\vec{v}$ is an eigenvector of A, then so is $-3\vec{v}$.

Solution: True, because if $A\vec{v} = \lambda\vec{v}$ then $A(-3\vec{v}) = -3A\vec{v} = -3\lambda\vec{v} = \lambda(-3\vec{v})$, so $-3\vec{v}$ is also an eigenvector with e.v. λ.

(d) **[1 point]** If $\lambda = 0$ is an eigenvalue of A then A is not invertible.

Solution: True: to say λ is an eigenvalue of A means that $det(A - \lambda I) = 0$. For $\lambda = 0$ this becomes $det(A) = 0$.