MAT 1341 Midterm Test #3 Solutions (Version 1)

October 20, 2014

(1) Let $B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$. Then the second row of B^{-1} is:

A. $[1\ 0\ -1]$
B. $[-1\ 0\ 1]$
C. $[0\ 1\ -1]$
D. $[2\ 0\ -1]$
E. $[1\ -1\ 0]$
F. None of the above.

Solution: You can find the inverse of B using row reduction; $B^{-1} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

Thus the correct answer is B.

(2) **[1 point]** If three $n \times n$ matrices A, B, C satisfy $AB - BA = C$ then ABA is:

A. $A^2B - C$ B. $A^2B - AC$ C. $BA^2 + CA$

D. A^2B E. $A^2B + AC$ F. $A^2B + BC$

Solution: If $AB - BA = C$, then $BA = AB - C$, and so

$$ABA = A(BA) = A(AB - C) = A^2B - AC$$

Hence the correct answer is B.

(3) **[1 point]** Let A be an $n \times n$ matrix. One of the following conditions is **not** equivalent to the others. Which one?

A. A is invertible.

B. The row-reduced echelon form A_R of A is the identity I_n.

C. The homogeneous system $A\vec{x} = \vec{0}$ has an infinite number of solutions.

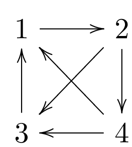
D. The system $A\vec{x} = \vec{b}$ is consistent for every choice of column vector $\vec{b}$.

E. There exists an $n \times n$ matrix C such that $AC = I_n$.

F. The rank of A is n.

Solution: This is literally a theorem stated in class; the correct answer is C, because if $A\vec{x} = 0$ has infinitely many solutions, then there must be at least one free variable, hence A must have less than n leading 1s, so its echelon form has a row of zeros.

(4) **[1 point]** Consider the following directed graph:



What is the adjacency matrix for this graph?

A. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ E. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ F. None of the above.

Solution: Recall that you have to put a "1" in row i, column j if there is an edge from j to i. This gives A as the correct answer.

(5) Suppose that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfies $T \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. What is $T \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$?

A. This can't be determined from the given information.

B. $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ C. $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ D. $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ E. $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ F. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Solution: Write $(-1, 7, -3)$ as a linear combination of $(1, 2, 0)$ and $(1, -1, 1)$:

$$\begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Then

$$T \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix} = 2T \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-3)T \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

(6) Consider the matrix $M = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) [**4 points**] Use row reduction to find the inverse of M.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{R_1 := R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{R_3 := \tfrac{1}{2}R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \underset{\sim}{R_2 := R_2 + 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

Hence $M^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

Marking: One point for augmenting M with the identity matrix; two points for correct reduction (subtract 1/2 point for minor errors, subtract 1 point for conceptual errors such as not applying the same steps on both sides of the augmentation or ending up with a matrix which is obviously not invertible). One point for the correct conclusion.

(b) [**2 points**] Use the inverse of M to solve the system:

$$\begin{cases} x + 2z &= 1 \\ -2x + y &= -2 \\ 2z &= 1 \end{cases}$$

Solution: Calculate

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ \frac{1}{2} \end{bmatrix}.$$

Hence the (unique) solution is $x = 0, y = -2, z = \frac{1}{2}$.

Marking: One point for knowing that you must calculate $M^{-1}\vec{b}$. One point for execution. (In case the wrong matrix was obtained in part (a), mark as if the answer given in (a) was correct.)

(7) Consider the following transformations T, S from $\mathbb{R}^2$ to $\mathbb{R}^2$: T is projection on the y-axis, and S is counterclockwise rotation by $\frac{3\pi}{4}$ around the origin.

(a) **[3 points]** Find the matrices for T and S. (You may assume that T, S are indeed linear transformations.)

Solution: We have $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$. Hence the matrix for T is $A = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$.

We have $S\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\sqrt{2}\\ \frac{1}{2}\sqrt{2}\end{bmatrix}$ and $S\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\sqrt{2}\\ -\frac{1}{2}\sqrt{2}\end{bmatrix}$. Hence the matrix for S is $B = \frac{1}{2}\sqrt{2}\begin{bmatrix}-1 & -1\\1 & -1\end{bmatrix}$.

Marking: 1/2 point for knowing that you must use the standard basis vectors. 1/2 point for knowing that the outcomes form the columns of the matrix. 1 point for correctly calculating A, 1 point for B. (Subtract 1/2 point for minor errors.)

(b) **[2 points]** Is it true that $T \circ S = S \circ T$? Explain why (not)?

Solution: No. Take for example the vector $\begin{bmatrix}1\\0\end{bmatrix}$. First applying T and then S gives $\begin{bmatrix}0\\0\end{bmatrix}$. On the other hand, first applying S and then T gives $\begin{bmatrix}0\\ \frac{1}{2}\sqrt{2}\end{bmatrix}$. These are different, so $TS \neq ST$.

Marking: One point for the idea, one for a correct example.

(c) **[1 points]** Find the matrix for the transformation which first rotates by $\frac{3\pi}{4}$ (counterclockwise) and then projects on the y-axis.

Solution: This is simply the product AB, so calculate $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\frac{1}{2}\sqrt{2}\begin{bmatrix}-1 & -1\\1 & -1\end{bmatrix} = \frac{1}{2}\sqrt{2}\begin{bmatrix}0 & 0\\1 & -1\end{bmatrix}$.

Marking: 1/2 point for the idea, one for the calculation. (Subtract 1/2 point if done in the wrong order.)

(8) For each of the following statements, say whether it is (always) true or (possibly) false. If true, explain in a few sentences why (referring to facts learned in class or from the book as needed). If false, give a counterexample (with numbers!).

Marking: In each part, give 1/2 point for the correct answer, 1/2 for the justification.

(a) **[1 point]** If A, B are invertible then so is $A+B$ (Here A, B are square matrices of the same size.)

Solution: False, let e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then A and B are invertible but $A + B$ is the zero matrix, which is not invertible.

(b) **[1 point]** If $AB = 0_n$, then $A = 0_n$ or $B = 0_n$. (Here, A, B are nxn matrices and 0_n is the nxn zero matrix.)

Solution: False, e.g. let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = 0$, but neither $A = 0$ nor $B = 0$.

(c) **[1 point]** If, in a 3x3 matrix, the third row is the sum of the first two, then the matrix is not invertible.

Solution: True: if the third row is the sum of the first two, then using row operations we can create a row of zeros. Hence the matrix doesn't have rank 3 and therefore is not invertible.

(d) **[1 point]** If B has a column of zeros and the product AB is defined, then AB has a column of zeros.

Solution: True: if the j-th column of B is 0, then for each i, the (i, j)-entry of AB is 0. Indeed, the (i, j)-entry is the dot product of the i-th row of A with the j-th column of B, so will be 0. Hence the j-th column of AB is 0.

SPACE FOR ROUGH WORK