MAT 1341 Midterm Test #2 Solutions (Version 1)

September 29, 2014

(1) **[1 point]** Consider a linear system with 100 equations and 99 variables. Can the system be inconsistent? Can it have a unique solution? Can it have infinitely many solutions?

A. Yes, Yes, Yes

B. Yes, Yes, No

C. Yes, No, Yes

D. No, Yes, Yes

E. No, No, Yes

F. Yes, No, No

Solution. A. (As discussed in class, the three possibilities can always arise when there are at least as many equations as variables.)

(2) **[1 point]** For which value(s) of a is the system below inconsistent?

$$\begin{cases} 2x - 4y &= 2 \\ x + ay &= 2 \end{cases}$$

A. $a = -1$

B. $a = -2$

C. $a \neq -1$

D. $a \neq -2$

E. For all values of a

F. For no values of a

Solution. The REF of the system is $\begin{bmatrix} 1 & -2 & | & 1 \\ 0 & a+2 & | & 1 \end{bmatrix}$. This is inconsistent precisely when $a = -2$. (For in that case, the bottom row would correspond to the equation 0=1.) For any other value of a, the system has a unique solution.

(3) **[1 point]** Suppose A is a 9x4 matrix, b is a column vector, and consider the system $Ax = b$. Which of the following statements is **not** correct?

A. If $Ax = b$ is consistent, then it has a unique solution.

B. $Ax = 0$ is consistent, but $Ax = b$ need not be consistent.

C. If $Ax = b$ has infinitely many solutions, then so does $Ax = 0$.

D. If $Ax = b$ is consistent, then $rank(A) = rank[A|b]$.

E. If $Ax = b$ is inconsistent, then $rank(A) < rank[A|b]$.

F. If $Ax = 0$ has infinitely many solutions, then so does $Ax = b$.

Solution. A: it is possible that the system is consistent and has infinitely many solutions, for example when all the rows are scalar multiples of each other.

(4) **[1 point]** Suppose that in a system with 3 equations and 3 variables the third equation is the sum of the first two. What can we conclude?

A. The system is inconsistent.

B. The system has a unique solution.

C. The system has infinitely many solutions.

D. If the system is consistent, then it has a unique solution.

E. If the system is consistent, then it has infinitely many solutions.

F. None of the above.

Solution. E: the system doesn't have to be consistent (the first two equations can represent parallel planes without a point in common), but if it is consistent, then the REF will have a row of 0s, and hence there will be a free variable giving infinitely many solutions.

(5) Consider the matrix $A = \begin{bmatrix} 0 & 0 & 3 & -1 \\ 2 & -4 & 5 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 4 & 1 & -3 \end{bmatrix}$.

(a) [**3 points**] Use the Gauss-Jordan algorithm to bring A into **reduced** REF. Clearly label each step by the row operation you use.

Solution. We have

$$\begin{bmatrix} 0 & 0 & 3 & -1 \\ 2 & -4 & 5 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 4 & 1 & -3 \end{bmatrix} \quad \overset{R_1 \leftrightarrow R_2}{\longrightarrow} \quad \begin{bmatrix} 1 & 2 & -3 & 0 \\ 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 2 \\ -1 & -2 & 3 & -5 \end{bmatrix} \quad \overset{\substack{R_2 := R_2 - 2R_1 \\ R_4 := R_4 + 2R_1}}{\longrightarrow}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & -1 \end{bmatrix} \quad \overset{\substack{R_3 := R_3 - R_2 \\ R_4 := R_4 - R_2}}{\longrightarrow} \quad \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \overset{R_2 := \frac{1}{3}R_2}{\longrightarrow}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \overset{R_1 := R_1 - R_2}{\longrightarrow} \quad \begin{bmatrix} 1 & -2 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Marking: Subtract 1 point for not labelling the operations (provided they are still correct). Subtract 1 point for incorrect use of an operation. Subtract 1 mark for stopping at REF. No loss of marks for minor arithmetic errors. Note: it's not necessary to use exactly the algorithm as described in the book, as long as RREF is obtained.

(b) [**1 point**] What is the rank of A?

Solution. There are 2 leading 1s, hence $rank(A) = 2$.

Marking: All or nothing. (Of course, if an error in part (a) was made leading to a different RREF, then consider that answer as correct for the present purposes; same in (c).

(c) [**2 points**] Use your answer from part (a) to find the solutions to the system $Ax = 0$.

Solution. Call the variables x, y, z, w. Then y and w are free, while x, z are leading. The solution becomes

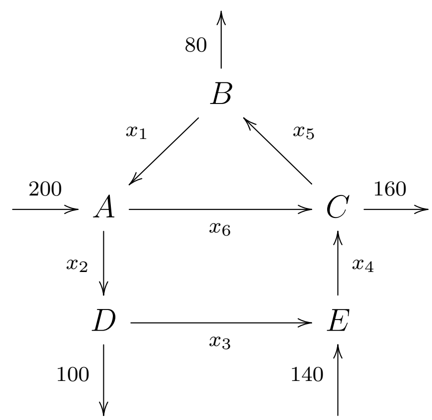
$$w = t, z = \frac{1}{3}w = \frac{1}{3}t, y = s, x = 2y - \frac{4}{3}w = 2s - \frac{4}{3}t.$$

This becomes

$$(x, y, z, w) = s(2, 1, 0, 0) + t(-\frac{4}{3}, 0, \frac{1}{3}, 1)$$

Marking: 1/2 point for correctly identifying free and leading variables. 1/2 point for showing that you need to use the RREF to read off the solutions. 1 point for correctly carrying this out. Subtract 1/2 point for each mistake in back substitution. (No loss of marks for omitting the last formulation of the general solution.)

(6) Consider the network of streets and intersections below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact number** of cars observed to enter or leave the intersections during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(a) [**3 points**] Write down a system of linear equations which describes the traffic flow. Also write down all the constraints on the variables $x_1, \ldots, x_6$. (Do not perform any operations on the equations; this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

Solution. The constraints are: each x_i is positive (one-way streets) and is an integer (no fractions of cars). The equations are:

Intersection	**In**	**Out**	**Equation**
A	$200 + x_1$	$x_2 + x_6$	$200 + x_1 = x_2 + x_6$
B	x_5	$x_1 + 80$	$x_5 = x_1 + 80$
C	$x_4 + x_6$	$x_5 + 160$	$x_4 + x_6 = x_5 + 160$
D	x_2	$x_3 + 100$	$x_2 = x_3 + 100$
E	$x_3 + 140$	x_4	$x_3 + 140 = x_4$

Marking: 1 point for showing that each intersection point gives an equation because the inflow has to equal the outflow. 1 point for correctly identifying the inflow and outflow at each point. Subtract 1/2 point for each inaccuracy. 1/2 point for each constraint.

(b) [**2 points**] The general solution to the system found in (a) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -80 \\ 120 \\ 20 \\ 160 \\ 0 \\ 0 \end{bmatrix}$$

If the road from A to C were closed due to roadwork, find the minimal flow along the road from D to E, **using the general solution of the system**. Justify your answer.

Solution. This road corresponds to x_6, so we get $x_6 = t = 0$. The general solution becomes

$$(x_1, x_2, x_3, x_4, x_5, x_6) = s(1, 1, 1, 1, 1, 0) + (-80, 120, 20, 160, 0, 0).$$

Since $x_1 \geq 0$, we find $s \geq 80$. For x_3, this implies that $x_3 = s+20 \geq 80+20 = 100$.

Marking: One point for showing that you understand that you must set $t = 0$, and simplify the general solution accordingly. One point for using the constraints correctly to find $x_3 \geq 100$. No marks if you don't use the general solution.

(7) For each of the following statements, say whether it is true or false. You don't have to motivate your answers.

(a) **[1 point]** If a linear system has more variables than equations, then it has infinitely many solutions.

Solution. False; the system may be inconsistent, e.g. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$.

(b) **[1 point]** A homogeneous system with more equations than variables only has the trivial solution.

Solution. False; consider for example $\begin{cases} x+y &= 0 \\ 2x+2y &= 0 \\ 3x+3y &= 0 \end{cases}$.

(c) **[1 point]** If A is a 3x3 matrix and b is a column vector for which $Ax = b$ is inconsistent, then $rank(A) < 3$.

Solution. True: to be inconsistent, we need $rank(A) < rank(A|b)$ (Rank Theorem). Since $rank(A|b)$ is at most 3 (because there are only 3 rows) it follows that $rank(A) < 3$.

Marking: No part marks, no justification needed.