Math 262 Equation Sheet

INFINITE SEQUENCES

Review

Let $\{a_n\}$ be a sequence. If $\lim\limits_{n\to+\infty} a_n = L$ then the sequence converges.

If $\lim\limits_{n\to+\infty} a_n$ results in either one of the following forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0^0, \infty^0, 1^\infty, \infty-\infty$ use L'Hopital's Rule to evaluate the limit.

L'Hoptial's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Properties of Sequences

Suppose that $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with $\lim\limits_{n\to+\infty} a_n = L$ and $\lim\limits_{n\to+\infty} b_n = M$ then the following results are true

Name	Property
Constant	$\lim\limits_{n\to+\infty} c = c$
Constant Multiple	$\lim\limits_{n\to+\infty} ca_n = c\lim\limits_{n\to+\infty} a_n = cL$
Additive	$\lim\limits_{n\to+\infty}(a_n+b_n) = \lim\limits_{n\to+\infty} a_n + \lim\limits_{n\to+\infty} b_n = L+M$
Difference	$\lim\limits_{n\to+\infty}(a_n-b_n) = \lim\limits_{n\to+\infty} a_n - \lim\limits_{n\to+\infty} b_n = L-M$
Product	$\lim\limits_{n\to+\infty}(a_n b_n) = \lim\limits_{n\to+\infty} a_n \bullet \lim\limits_{n\to+\infty} b_n = LM$
Quotient	$\lim\limits_{n\to+\infty}\frac{a_n}{bn} = \frac{\lim\limits_{n\to+\infty} a_n}{\lim\limits_{n\to+\infty} b_n} = \frac{L}{M}$ provided that $M \neq 0$
Squeeze Theorem For Sequences	Let $\{a_n\},\{b_n\}$, and $\{c_n\}$ be sequences such that $\{a_n\} \le \{c_n\} \le \{b_n\}$. If $\lim\limits_{n\to+\infty} a_n = L$ and $\lim\limits_{n\to+\infty} b_n = L \Rightarrow \lim\limits_{n\to+\infty} c_n = L$

TESTS TO DETERMINE THE CONVERGENCE/DIVERGENCE OF A SERIES

Name	What Does It Say
Integral Test	$\sum a_k$ is a series of positive terms and $f(x) = a_k$. If the function is decreasing and continuous on $[a,+\infty)$, then the series converges if $\int_a^{+\infty} f(x)dx$ is defined. The series diverges if $\int_a^{+\infty} f(x)dx = \pm\infty$
Divergence Test	$\sum a_k$ diverges if $\lim\limits_{k\to+\infty} a_k \neq 0$
Ratio Test	$\sum a_k$ is a series of positive terms with $p = \lim\limits_{k\to+\infty}\frac{a_{k+1}}{a_k}$ If $p>1 \Rightarrow$ the series diverges. If $p<1 \Rightarrow$ the series converges. If $p=1 \Rightarrow$ no conclusion.
Root Test	$\sum a_k$ is a series of positive terms with $p = \lim\limits_{k\to+\infty}\sqrt[k]{a_k}$ If $p>1 \Rightarrow$ the series diverges. If $p<1 \Rightarrow$ the series converges. If $p=1 \Rightarrow$ no conclusion.
Comparison Test	Suppose that $\sum_1^\infty a_k$ and $\sum_1^\infty b_k$ are two series with positive terms such that $a_k \le b_k$ for all k. If $\sum_1^\infty b_k$ converges, then $\sum_1^\infty a_k$ converges. If $\sum_1^\infty a_k$ diverges, then also $\sum_1^\infty b_k$ diverges.
Limit Comparison Test	Suppose that $\sum_1^\infty a_k$ and $\sum_1^\infty b_k$ are two series with positive terms with $p = \lim\limits_{k\to+\infty}\frac{a_k}{b_k}$ If $p>0 \Rightarrow$ either both series converge, or both series diverge.
Name	What Does It Say
Alternating Series Test	Suppose that $a_k > 0$ for $k = 1,2,3,\ldots$ then the series has the form of either: $a_1 - a_2 + a_3 - a_4 + \ldots$ or $-a_1 + a_2 - a_3 + a_4 - \ldots$. The series converges if and only if 1. $a_{k+1} \le a_k$; the series is decreasing, and 2. $\lim\limits_{k\to+\infty} a_k = 0$
Ratio Test Absolute Convergence	$\sum a_k$ is a series of nonzero terms with $p = \lim\limits_{k\to+\infty}\left\|\frac{a_{k+1}}{a_k}\right\|$ If $p>1 \Rightarrow$ the series diverges. If $p<1 \Rightarrow$ the series converges. If $p=1 \Rightarrow$ no conclusion.

POWER OF SERIES AND RADIUS OF CONVERGENCE

Form	Sigma Notation	Radius of Convergence
$\frac{1}{1-x} = 1+x+x^2+x^3+x^4.+\ldots$	$=\sum_1^\infty x^n$	$-1\le x\le 1$
$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots$	$=\sum_1^\infty nx^{n-1}$	$-1\le x\le 1$
$\ln(x+1) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$	$=\sum_1^\infty(-1)^n\frac{x^n}{n}$	$-1\le x\le 1$
$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$	$=\sum_1^\infty(-1)^n\frac{x^{2n+1}}{2n+1}$	$-1\le x\le 1$
$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$	$=\sum_1^\infty\frac{x^n}{n!}$	$(-\infty,\infty)$
$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}\ldots$	$=\sum_1^\infty(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\ldots$	$=\sum_1^\infty(-1)^n\frac{x^{2n}}{(2n)!}$	$(-\infty,\infty)$

VECTORS AND COORDINATE GEOMETRY

Vectors & Vector Dot Products

Let $\vec{u}$ be a vector having n components, then the length or norm of is $\vec{u}$ defined to be

$$|\vec{u}| = \sqrt{(n_1)^2+(n_2)^2+(n_3)^2+\ldots+(n_n)^2}$$

Let $\vec{u}$ and $\vec{v}$ be two vectors, then their dot product is defined to be:

$$\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$$

Properties

Symmetry	$\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$
Multiplication with a scalar	$(k\vec{u})\cdot\vec{v}=k(\vec{u}\cdot\vec{v})$ k∈R
Linearity	$\vec{u}(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

Uses of Dot Product

A. Show orthoganality: $\vec{u}\cdot\vec{v}=0 \Leftrightarrow \vec{u}\perp\vec{v}$
B. Finding the angle between two vectors:

$$\theta=\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

Unit Vectors

Suppose that $\vec{u}$ is a vector having n components, then its unit vector is found be dividing each component of the vector by its norm. That is,

$$|\vec{u}|_{unit}=\frac{1}{|\vec{u}|}\bar{u}=\frac{1}{\sqrt{(n_1)^2+(n_2)^2+(n_3)^2+...+(n_n)^2}}\vec{u}$$

VECTOR VALUED FUNCTIONS

Tangent Directions

If $\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$ is a position vector to the points on the curve $(x(t),y(t),z(t))$ then we can find the following quantities

Average Velocity	$v_{avg}=\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$, Δt = change in time
Velocity	$v(t)=r'(t)=\frac{dr}{dt}=\frac{dx}{dt}[x(t)]i+\frac{dy}{dt}[y(t)]j+\frac{dz}{dt}[z(t)]k$
Speed	$\lvert v(t)\rvert$
Acceleration	$a(t)=r''(t)=\frac{d^2r}{dt^2}=\frac{dx}{dt}[x'(t)]i+\frac{dy}{dy}[y'(t)]j+\frac{dz}{dt}[z'(t)]k$

Arc – length

$$s=s(t)=\int_{t_0}^{t}|\vec{r}'(t)|dt$$

Formulas for Curvature, Torsion, Radius of Curvature

Name	Formula
Unit Tangent Vector	$\bar{T}=\frac{\bar{v}}{\lvert\bar{v}\rvert}$
Curvature	$\kappa=\frac{\lvert\bar{v}\times\bar{a}\rvert}{\lvert v\rvert^3}$
Radius of Curvature	$\rho=\frac{1}{\kappa}$
Binormal Vector	$\bar{B}=\frac{\bar{v}\times\bar{a}}{\lvert\bar{v}\times\bar{a}\rvert}$
Unit Normal Vector	$\bar{N}=\bar{B}\times\bar{T}=\frac{\frac{d\bar{T}}{dt}}{\left\lvert\frac{d\bar{T}}{dt}\right\rvert}$
Tangential and Normal Components	$a=\frac{dv}{dt}\bar{T}+\lvert v\rvert^2\kappa\bar{N}$
Torsion	$\tau=\frac{(v\times a)\frac{da}{dt}}{\lvert v\times a\rvert^2}$

Frenet – Serret Formulas

$\frac{d\bar{T}}{ds}=\kappa\bar{N}$	$\frac{d\bar{B}}{ds}=-\tau\bar{N}$	$\frac{d\bar{N}}{ds}=-\kappa\bar{T}+\tau\bar{B}$

PARTIAL DIFFERENTIATION

Definition of Limit on a plane

$$\lim_{(x,y)\to(a,b)}f(x,y)=L$$

Implies that

A. Every neighborhood of (a, b) contains points of the domain of $f(x,y)$ different from (a, b).
B. For every ε (positive) there exists a positive δ such that $|f(x,y)-L|<\varepsilon$ whenever (x, y) is in the domain of f and satisfies the inequality: $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$

Definition of Continuity on a Plane

For continuity to happen, the following 2 conditions must hold:

1. $\lim_{(x,y)\to(a,b)}f(x,y)$ exists
2. $\lim_{(x,y)\to(a,b)}f(x,y)=f(a,b)$

Partial Derivatives

Given a function $f(x,y)$, the partial with respect to x, denoted as f_x, at the point (a,b) is defined to be

$$f_x(a,b)=\lim_{h\to 0}\frac{f(a+h,b)-f(a)}{h}$$

Similarly the partial with respect to y, at the point (a,b), denoted as f_y, is defined to be

$$f_y=\lim_{k\to 0}\frac{f(a,b+k)-f(b)}{k}$$

In practice we use the following

$f_x=f_1=\frac{\partial f}{\partial x}$	$\Rightarrow$	differentiate the function f with respect to x while holding all other variables as constants.
$f_y=f_2=\frac{\partial f}{\partial y}$	$\Rightarrow$	differentiate the function f with respect to y while holding all other variables as constants.

The idea can easily be extended to functions having more than 2 variables.

Higher Derivatives

Second derivatives for functions of 2 variables

$f_{xx}=f_{11}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$	Second derivative of f with respect to x
$f_{yy}=f_{22}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$	Second derivative of f with respect to y
$f_{xy}=f_{12}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$	Mixed partial
$f_{yx}=f_{21}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$	Mixed partial

Chain Rule

$z=f(x,y)$ is a differentiable function of x and y with $x=x(t)$ and $y=y(t)$ then

$$z'(t)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: $z(x, y) = x \sin y$ where $x(t) = e^t$ and $y(t) = t^2$

Solution:

$$z'(t) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial t} = e^t \qquad \frac{\partial y}{\partial t} = 2t$$

$$\frac{\partial z}{\partial x} = \sin y \qquad \frac{\partial z}{\partial y} = x \cos y$$

$$z'(t) = e^t \sin y + 2tx \cos y$$

The chain rule can be extended to include functions which have more than one variable.

Let z be a differentiable function of n variables, $x_1, x_2, x_3, \ldots, x_n$ and each x_j is a differentiable function of m variables, $t_1, t_2, t_3, \ldots, t_m$ then z is ultimately a function of $t_1, t_2, t_3, \ldots, t_m$. So

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Tangent Planes & Gradients

Suppose that $z = f(x, y)$ has tangent planes everywhere, then the gradient is:

$$\nabla f(x, y) = \hat{i}\frac{\partial}{\partial x}(x, y) + \hat{j}\frac{\partial}{\partial y}(x, y)$$

$\nabla f(x, y)$ = Direction of the steepest slope upwards in the XY plane

$-\nabla f(x, y)$ = Direction of the steepest slope downwards in the XY plane.

$\|\nabla f(x, y)\|$ = Slope in steepest direction

$D_u f = \hat{u} \cdot \nabla f$ = Directional derivative of f with respect to $\hat{u}$

$\hat{u} + \hat{k} D_u f$ = Tangent vector with respect to $\hat{u}$ direction

$\hat{k} - \nabla f$ = Upward normal vector to the surface
= Perpendicular to all tangent vectors
= Normal vector to the tangent plane.

LOCAL EXTREMAS

Suppose that the surface $z = f(x, y)$ has tangent planes at every point. The points (x_0, y_0, z_0) on the surface where the tangent plane is horizontal

$\Rightarrow \overline{\nabla} f \Big|_{(x_0, y_0)} = 0$, are called critical point (local extrema).

$D_u^2 f$ positive for all θ $\Rightarrow$ nearby slopes are up
$\Rightarrow$ surface bends up
$\Rightarrow$ local minimum

$D_u^2 f$ negative for all θ $\Rightarrow$ nearby slopes are down
$\Rightarrow$ surface bends down
$\Rightarrow$ local maximum

$D_u^2 f$ changes slope $\Rightarrow$ nearby slopes is up in some directions and down in other directions
$\Rightarrow$ surface bends up and down
$\Rightarrow$ saddle

The test can be expressed in terms of the Jacobian matrix

$$D = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $\det(D) < 0$ $\Rightarrow$ we have a saddle point

If $\det(D) < 0$ and $a + c > 0$ $\Rightarrow$ we have a local minimum
If $\det(D) < 0$ and $a + c < 0$ $\Rightarrow$ we have a local maximum

Lagrange Multipliers
To find the extreme values:

1. First get it into a Lagrangian function: $L(x, y) = f(x, y) + \lambda g(x, y)$
2. Solve for the critical point by finding out when
$$0 = \frac{\partial L}{\partial x} = f_x(x, y) + \lambda g(x, y), \quad 0 = \frac{\partial L}{\partial y} = f_y(x, y) + \lambda g(x, y), \quad 0 = \frac{\partial L}{\partial \lambda} g(x, y)$$
3. Once the critical points have been found, plug it back into the function and test to see which point satisfies the given criteria.