University of Ottawa

Department of Mathematics and Statistics

MAT 1302C : Mathematical Methods II

Professor: Dr. Hua

First Midterm Exam – Version B

Oct 2, 2014, Thursday

Surname ______________________ First Name ______________________

Student # ______________________

DGD # (1=Monday; 2=Thursday)__________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You have to show your work for each question.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	5	5	5	5	5	5	30
Grade							

1. Multiple choice questions, no partial marks, chose the best possible answer.

(a) (1 point) $\begin{bmatrix} 4 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, (b) $\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$, (c) $\begin{bmatrix} 6 \\ 15 \end{bmatrix}$, (d) $\begin{bmatrix} 4 \\ 10 \\ 18 \end{bmatrix}$, (e) $\begin{bmatrix} 15 \\ 6 \end{bmatrix}$.

Solution: (e)

(b) (1 point) Given $\vec{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Which of them are in the span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$?

(a) $\vec{u}$ and $\vec{v}$, (b) $\vec{u}$ and $\vec{w}$, (c) $\vec{w}$ and $\vec{v}$, (d) $\vec{u}$ only, (e) $\vec{w}$ only.

Solution: (c)

(c) (3 points) Given

$$A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(i) List all matrices which are in reduced row echelon form;

(a) A only, (b) B only, (c) C only, (d) D only, (e) B and C.

Solution: (b)

(ii) List all matrices which are in row echelon form but not in reduced row echelon form;

(a) A only, (b) B only, (c) C only, (d) D only, (e) A and C.

Solution: (d)

(iii) List all matrices which are not in row echelon form.

(a) A only, (b) B only, (c) C only, (d) D only, (e) A and C.

Solution: (e)

2. Consider the following linear system

$$\begin{cases} 4x_1 + 8x_2 - 4x_3 + 20x_4 = 4 \\ 2x_1 + 2x_2 - 2x_3 + 6x_4 = 3 \\ x_1 + x_2 - x_3 + 3x_4 = 2 \end{cases}$$

(a) [**4 points**] Row reduce the augmented matrix to the reduced row echelon form.

(b) [**1 point**] Determine if the linear system is consistent or inconsistent. (If the system is consistent, you do not need to find the general solution.)

Solution: (a)

We write the augmented matrix of the system and row reduce it

$$\left[\begin{array}{cccc|c} 4 & 8 & -4 & 20 & 4 \\ 2 & 2 & -2 & 6 & 3 \\ 1 & 1 & -1 & 3 & 2 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{4}R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 2 & 2 & -2 & 6 & 3 \\ 1 & 1 & -1 & 3 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & -1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0.5 \end{array}\right] \xrightarrow{R_3 \to 2R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 0 \\ 0 & -2 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(b) Since the rightmost column is a pivot column, the linear system is inconsistent.

3. [**5 points**] Determine all values of h such that the vector $\begin{bmatrix} -1 \\ h \\ 6 \end{bmatrix}$ belongs to Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \right\}$.

Solution: We write the corresponding augmented matrix and then row reduce it:

$$\left[\begin{array}{cc|c} 1 & 3 & -1 \\ -1 & 1 & h \\ 2 & -2 & 6 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 4 & h-1 \\ 0 & -8 & 8 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 4 & h-1 \\ 0 & 0 & 2h+6 \end{array}\right]$$

The vector $\begin{bmatrix} -1 \\ h \\ 6 \end{bmatrix}$ belongs to Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -8 \end{bmatrix} \right\}$ if and only if the system with the latter matrix as the augmented matrix is consistent, that is, the rightmost column is not a pivot column. This is equivalent to $2h + 6 = 0$, or $h = -3$.

4. [**5 points**] Find the general solution to the matrix equation $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & -3 & -1 & 1 \\ -2 & 6 & 3 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Indicate which variables are basic, which variables are free.

Solution: We row reduce the augmented matrix of the corresponding linear system.

$$\left[\begin{array}{cccc|c} 1 & -3 & -1 & 1 & 3 \\ -2 & 6 & 3 & 1 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & -3 & -1 & 1 & 3 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 7 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right]$$

x_1, x_3 basic. x_2, x_4 free. Therefore the general solution is:

$$\begin{cases} x_1 = 3x_2 - 4x_4 + 7 \\ x_2 = \text{free} \\ x_3 = -3x_4 + 4 \\ x_4 = \text{free} \end{cases}$$

5. [**5 points**] Consider the following linear system

$$\begin{cases} 3x_1 + x_2 &= 2x_1 + 2x_2 + 4 \\ 2x_1 + hx_2 &= k. \end{cases}$$

(i) For what values of h and k does the system have no solution?

(ii) For what values of h and k does the system have only one solution?

(iii) For what values of h and k does the system have infinitely many solution?

Solution: Rewrite the system as

$$\begin{cases} x_1 - x_2 &= 4 \\ 2x_1 + hx_2 &= k. \end{cases}$$

$$[A|\vec{b}] = \begin{bmatrix} 1 & -1 & 4 \\ 2 & h & k \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 4 \\ 0 & h+2 & k-8 \end{bmatrix}.$$

(i) $h = -2, k \neq 8$;

(ii) $h \neq -2, k \in \mathbb{R}$;

(iii) $h = -2, k = 8$;

6. [**5 points**] Given four vectors $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ -17 \\ 9 \end{bmatrix}$. Write $\vec{y}$ as a linear combination of $\vec{u}, \vec{v}, \vec{w}$.

Solution:

$$[\vec{u} \quad \vec{v} \quad \vec{w} \quad \vec{y}] = \begin{bmatrix} 2 & 0 & 6 & 2 \\ -1 & 8 & 5 & -17 \\ 1 & -2 & 5 & 9 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 3 & 1 \\ -1 & 8 & 5 & -17 \\ 1 & -2 & 5 & 9 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1, R_3 - R_1} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 8 & 8 & -16 \\ 0 & -2 & 2 & 8 \end{bmatrix} \xrightarrow{\frac{1}{8}R_2} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & 2 & 8 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_3, R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore, $\vec{y}$ can be written as a linear combination of $\vec{u}, \vec{v}, \vec{w}$:

$$\vec{y} = -2\vec{u} - 3\vec{v} + \vec{w}.$$