MAT 1341 Assignment 4 - DGD#_____

Summer 2014

Total: 14 points; Due: July 14, **beginning of DGD**

Family Name:______________________

First Name:______________________

Student Number: __________________

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY

1. For each question, explain how you arrive at your answer. You earn points by correctly using mathematical notation, using correct reasoning and logic, and by identifying the key aspects of the problems.

2. You are allowed to discuss the problems with your classmates, but the work you hand in should be your own. Copying other people's solutions counts as plagiarism and will be dealt with as such.

1. **[1 point]** Let A be 2×2 invertible matrix, and let 2 be an eigenvalue of A.

One eigenvalue of $A^3 + 2A^{-1}$ is ________.

Solution: An eigenvalue of $A^3 + 2A^{-1}$ is $(2)^3 + 2/2 = 9$.

2. **[2 points]** Let $S = \{\vec{u}, \vec{v}, \vec{w}\}$ be a subset set of $\mathbb{R}^5$, such that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} = \vec{w} \cdot \vec{v} = \vec{0}$. Show that the set S is linearly independent.

Solution: We need to set up the equation

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}, \Rightarrow$$

$$(c_1\vec{u} + c_2\vec{v} + c_3\vec{w}) \cdot \vec{u} = \vec{0} \cdot \vec{u} = 0, \Rightarrow$$

$$c_1||\vec{u}||^2 = 0, \Rightarrow c_1 = 0.$$

Similarly we can imply that $c_2 = 0$, $c_3 = 0$. Thus S is linearly independent.

Marking: 0.5 points for setting up the equation, 1.5 points for deducing $c_1 = c_2 = c_3 = 0$.

3. **[4 points]** Let $A = \begin{bmatrix} 2 & 4 & 3 & 9 \\ 1 & 2 & 1 & 4 \\ -3 & -6 & -1 & -10 \\ -1 & -2 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Give the rank of the matrix A.

   (b) Give the dimension for each of the following subspaces row(A), col(A) and null(A).

   (c) Give a basis for each of the following subspaces row(A), col(A) and null(A).

Solution:

(a) The rank of A is 2.

(b) $\dim(\text{row}(A)) = \text{rank}(A) = 2$, $\dim(\text{col}(A)) = \text{rank}(A) = 2$ and

$$\dim(\text{null}(A)) = \#\text{ of columns } - \text{rank}(A) = 4 - 2 = 2.$$

(c) A basis for row(A) is $\{(1,2,0,3),(0,0,1,1)\}$. A basis for col(A)

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

Solving $A\mathbf{x} = \mathbf{0}$, we get the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

A basis for null(A) is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Marking: 0.5 points for (a); 1.5 points for (b); 1.5 points for (c).

4. [**3 points**] Consider the following independent set $S = \{X_1,\ X_2,\ X_3\}$ of vectors from $\mathbb{R}^4$:

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}, X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix}.$$

Use the Gram-Schmidt algorithm to convert the set $S = \{X_1,\ X_2,\ X_3\}$ into an orthogonal set $B = \{F_1,\ F_2,\ F_3\}$.

Solution: Let $F_1 = X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Then

$$F_2 = X_2 - \frac{X_2 \cdot F_1}{||F_1||^2} F_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{||F_1||^2} F_1 - \frac{X_3 \cdot F_2}{||F_2||^2} F_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Marking: 0.5+1+1.5=3 points

5. [**2 points**] Let U be a subspace of $\mathbb{R}^4$ and let the set $S = \{X_1,\ X_2,\ X_3\}$ be an orthogonal basis of U, where

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Given $X = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 42 \end{bmatrix}$. Find the orthogonal projection of X onto U, i.e., the best approximation in U to the vector X.

(b) Compute the approximation error $||X - proj_U X||$.

Solution: (a) The orthogonal projection of X onto U is

$$proj_U X = \frac{X \cdot X_1}{X_1 \cdot X_1} X_1 + \frac{X \cdot X_2}{X_2 \cdot X_2} X_2 + \frac{X \cdot X_3}{X_3 \cdot X_3} X_3 = \begin{bmatrix} 1 \\ 7 \\ 20 \\ 20 \end{bmatrix}.$$

(b)

$$X - proj_U X = \begin{bmatrix} 2 \\ -6 \\ -20 \\ 22 \end{bmatrix}.$$

The approximation error $||X - W|| = \sqrt{2^2 + (-6)^2 + (-20)^2 + 22^2} = \sqrt{924}$.

Marking: 1+1=2 points .

6. **[2 points]** Let $\mathbb{M}_2$ denote the set of all 2×2 matrices. We define addition with the standard addition of matrices, but with scalar multiplication given by

$$k \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & b \\ c & kd \end{bmatrix},$$

where k is a scalar.

Is $(k+s) \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = k \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} + s \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for scalars k, s? Explain.

Solution: No! Since $(k+s) \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (k+s)a & b \\ c & (k+s)d \end{bmatrix}$, and

$$k \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} + s \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (k+s)a & 2b \\ 2c & (k+s)d \end{bmatrix}$$

Marking: 0.5 points for correct conclusion; 1.5 points for explanation.

7. **[Bonus: 1 point]** Is the set $\{\cos^2 x, 3, \sin^2 x\}$ linearly independent?

Solution: No. Since $\cos^2 x + \sin^2 x = 1$.