MAT 1341 Assignment 3 - DGD#_____

Summer 2014

Total: 14 points; Due: June 30, **beginning of DGD**

Family Name:______________________

First Name:_______________________

Student Number: __________________

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY

1. For each question, explain how you arrive at your answer. You earn points by correctly using mathematical notation, using correct reasoning and logic, and by identifying the key aspects of the problems.

2. You are allowed to discuss the problems with your classmates, but the work you hand in should be your own. Copying other people's solutions counts as plagiarism and will be dealt with as such.

1. **[1 point]** Suppose A is a 4×4 matrix with $detA = 2$. Find $det(\frac{1}{2}A^TA^7I_4A^TA^{-1})$.

Solution: $det(\frac{1}{2}A^TA^7I_4A^TA^{-1}) = (\frac{1}{2})^4(2)(2^7)(1)(2)(\frac{1}{2}) = 16$.

2. **[2 points]** Using row reduction and the effect of row operations on the determinant, compute the determinant of the matrix $A = \begin{bmatrix} 0 & 1 & 4 & 7 \\ 2 & 4 & -6 & 8 \\ 1 & 2 & -1 & 12 \\ 2 & 5 & 2 & 15 \end{bmatrix}$

Solution:

$$\det A = \begin{vmatrix} 0 & 1 & 4 & 7 \\ 2 & 4 & -6 & 8 \\ 1 & 2 & -1 & 12 \\ 2 & 5 & 2 & 15 \end{vmatrix} \overset{R_1 \leftrightarrow R_3}{=\!=\!=\!=\!=} - \begin{vmatrix} 1 & 2 & -1 & 12 \\ 2 & 4 & -6 & 8 \\ 0 & 1 & 4 & 7 \\ 2 & 5 & 2 & 15 \end{vmatrix}$$

$$\overset{\substack{R_2 \to R_2 - 2R_1 \\ R_4 \to R_4 - 2R_1}}{=\!=\!=\!=\!=\!=\!=} - \begin{vmatrix} 1 & 2 & -1 & 12 \\ 0 & 0 & -4 & -16 \\ 0 & 1 & 4 & 7 \\ 0 & 1 & 4 & -9 \end{vmatrix} \overset{R_4 \to R_4 - R_3}{=\!=\!=\!=\!=\!=\!=} - \begin{vmatrix} 1 & 2 & -1 & 12 \\ 0 & 0 & -4 & -16 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & -16 \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{=\!=\!=\!=\!=} \begin{vmatrix} 1 & 2 & -1 & 12 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & -16 \end{vmatrix} = 1(1)(-4)(-16) = 64.$$

Marking: 1.5 points for correct row reduction, 0.5 piints the correct conclusion.

3. **[2 points, 1 point each]** For each of the following, give an example if it exists. If it doesn't exist, explain why.

(a) A 3×3 matrix whose only eigenvalue is 0 but whose rank is non-zero.

(b) A 3×3 matrix A with $A^6 + I_3 = 0$. (Hint: use the determinant.)

Solution: (a) **Exists!** For example, $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) **Does not exist!** In fact, if there is such A, then

$$\det(A^6) = \det(-I_3) = -1.$$

Since $x^6 = -1$ has no real solutions, the assumption that A exists leads to a contradiction.

4. [**1 point**] Consider the matrix $A = \begin{bmatrix} 1 & c & 4 \\ 0 & 1 & c \\ 1 & 1 & c \end{bmatrix}$. (Here, c is a real number.) Using determinants, find all values of c for which the matrix is invertible.

Solution: $\det(A) = c^2 - 4 \neq 0, \Rightarrow c \neq \pm 2$.

5. [**1 point**] Use the inversion algorithm to find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. **[4 points=3+1]** Let $A = \begin{bmatrix} 7 & -8 & 0 \\ 4 & -5 & 0 \\ -1 & 1 & 1 \end{bmatrix}$. The characteristic polynomial of A $c_A(\lambda) = (1-\lambda)(\lambda-3)(\lambda+1)$.

(a) For each eigenvalue of A, find the corresponding eigenvectors.

(b) If possible, find a diagonal matrix C and an invertible matrix B such that $A = BCB^{-1}$. If this is not possible, explain why.

Solution: (a) The eigenvalues of A are 1, 3, -1 (Each has multiplicity 1).

When $\lambda = 1$, we solve $(I - A)\vec{x} = 0$:

$$[I - A|\vec{0}] = \begin{bmatrix} -6 & 8 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 = 0$, $x_2 = 0$. Thus $(I - A)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

When $\lambda = 3$, we solve $(3I - A)\vec{x} = 0$:

$$[3I - A|\vec{0}] = \begin{bmatrix} -4 & 8 & 0 & 0 \\ -4 & 8 & 0 & 0 \\ 1 & -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 + 4x_3 = 0$, $x_2 + 2x_3 = 0$. Thus $(3I - A)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}.$$

When $\lambda = -1$, we solve $(I + A)\vec{x} = 0$:

$$[A + I|\vec{0}] = \begin{bmatrix} -8 & 8 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 1 & -1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 - x_2 = 0$, $x_3 = 0$. Thus $(I + A)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(b) Since A has three basic eigenvectors, e.g.,

$$\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

A is diagonalizable. For example,

$$B = [\vec{u} \;\; \vec{v} \;\; \vec{w}] = \begin{bmatrix} 0 & -4 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

7. **[3 points = 2+1]** Let $\vec{w} = (1, 3, 5)$ and $U = \{\vec{u} \in \mathbb{R}^3 | \vec{w} \times \vec{u} = \vec{0} = (0, 0, 0)\}$.

(a) Show that U is a subspace of $\mathbb{R}^3$.

(b) Find a spanning set for U.

Solution: (a) Proof. (1) $\vec{0} \in U$, since $\vec{w} \times \vec{0} = (0, 0, 0)$.

(2) U is closed under addition: Let $\vec{u_1}, \vec{u_2} \in U$, i.e., $\vec{w} \times \vec{u_1} = \vec{0}$, $\vec{w} \times \vec{u_2} = \vec{0}$. Then

$$\vec{w} \times (\vec{u_1} + \vec{u_2}) = \vec{w} \times \vec{u_1} + \vec{w} \times \vec{u_2} = \vec{0}, \Rightarrow \vec{u_1} + \vec{u_2} \in U.$$

(3) U is closed under scalar multiplication: Let $\vec{u}, \in U$, i.e., $\vec{w} \times \vec{u} = \vec{0}$, let $k \in \mathbb{R}$. Then

$$\vec{w} \times (k\vec{u}) = k\vec{w} \times (\vec{u}) = \vec{0}, \Rightarrow k\vec{u} \in U.$$

Thus U is a subspace.

(b) Let $\vec{u} = (x, y, z)$. Then $\vec{w} \times \vec{u} = (0, 0, 0) \Rightarrow 3z - 5y = 0, 5x - z = 0, y - 3x = 0, \Rightarrow$

$$(x, y, z) = x(1, 3, 5) = x\vec{w}.$$

Thus

$$U = span\{\vec{w}\}.$$