CSI3105 Fall 2011
Assignment 2 Solutions

1. We will use the log relation $\lg n = (\lg e)^* (\ln n)$ .....(*)
and $\log ab = \log a + \log b$ (**)
and L'Hôpital's Rule .....(LH)

a) Let $g(n) = n\lg n$, let $f(n) = kn\lg(kn)$, $k>0$. (NOTE: $k>=0$ in the assignment was a typo, as mentioned in class)

$$\lim_{n\to\infty} \frac{g(n)}{f(n)}$$

$$= \lim_{n\to\infty} \frac{n\lg n}{kn\lg kn}$$

$$= \lim_{n\to\infty} \frac{\lg n}{k\lg kn}$$ .....simplifying

$$= \lim_{n\to\infty} \frac{(\lg e)(\ln n)}{(k)(\lg k + \lg n)}$$ .....using (**)

$$= \lim_{n\to\infty} \frac{(\lg e)(\ln n)}{(k\lg k) + k\lg e(\ln n)}$$ .....using (*)

$$= \lim_{n\to\infty} \frac{(\lg e)/n}{(k\lg e)/n}$$ .....using (LH)

$$= \lim_{n\to\infty} \frac{1}{k}.$$ .....simplifying

$= 1/k$. Since $1/k$ is a constant which is >0, we have $g(n)$ is $\Theta(f(n))$ as required.

b) Let $g(n) = 6n^2 + 20n$, let $f(n) = n + 5$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)}$$

$$= \lim_{n\to\infty} \frac{6n^2 + 20n}{n + 5}$$

$$= \lim_{n\to\infty} \frac{12n + 20}{1}$$ .....using (LH)

$= \infty$. Thus $g(n)$ is $\Omega(f(n))$ as required.

2.
Average Case Analysis:
Input size: 4
Basic Operation Counted: Comparison of x with keys in S.

Consider all possible input lists S of size 4, and partition those inputs as follows: For i = 1, 2, 3, 4 let I_i be the set of all inputs for which the number x appears in position i.
For the inputs for which x does not appear in the list, we let:
I_5, I_6, I_7, be the set of all inputs for which the number x lies in the gap between S[1] and S[2], S[2] and S[3] and S[3] and S[4].
Let, I_8 be the set of all inputs for which the number x is smaller than all the numbers in S, and let I_9 be the set of all inputs for which the number x is larger than all the numbers in S.

All input classes are equally likely, so $p(I_i) = 1/9$ for all i.
The following are the values for $t(I_i)$, found by tracing the algorithm:

i	1	2	3	4	5	6	7	8	9
t(Ii)	7	5	3	1	8	6	4	8	2

So $A(4) = \sum t(I_i)p(I_i)$: i = 1 to 9)
$= 1/9 (\sum t(I_i)$: i = 1 to 9)
$= 1/9$ (44)
$= 4.89$

3. $W(n) = 1$ for n=1, 2 or 4
For n>4, n a power of 2, $W(n) = W(n/2) + n/2 + 1$.
So, for $n = 2^k$, k>=3, $W(2^k) = W(2^{k-1}) + 2^{k-1} + 1$.

$W(2^k) = W(2^{k-1}) + 2^{k-1} + 1$ (using back substitution)
$= W(2^{k-2}) + 2^{k-2} + 2^{k-1} + 1 + 1$
(line 3) $= W(2^{k-3}) + 2^{k-3} + 2^{k-2} + 2^{k-1} + 1 + 1 + 1$
.
:
(line i) $= W(2^{k-i}) + 2^{k-i} + 2^{k-i+1} + \ldots + 2^{k-1} + i$

:
= ? ***Here we must decide at which line we reach the base case.

The possible base cases for us to hit in the back substitution are n=1, 2 or 4. If we started with $n=2^k$, then we hit the base case for n=4 first in the back substitution. This happens when $W(2^{k-i}) = W(4)$, i.e. when k-i = 4, which implies line i=(k-2). In this case, we have, in the (k-2) line of the back substitution (i.e. substituting k-2 for i in line i):

$= W(4) + 2^{k-(k-2)} + 2^{k-(k-2)+1} + \ldots + 2^{k-1} + (k-2)$
$= 1 + 2^2 + 2^3 \ \ldots + 2^{k-1} + k - 2$
$= 1 + \sum (2^i : i= 2 \text{ to } k-1) + k - 2$
$= 1 + \sum (2^i : i= 0 \text{ to } k-1) - (2^0 + 2^1) + k - 2$
$= 1 + (2^k - 1) - 1 - 2 + k - 2$ (using the formula $\sum (2^i : i= 0 \text{ to } n) = (2^{n+1} - 1)$ from App. A)
$= 2^k + k - 5$ (simplifying)

Since k=lgn, this give us
$W(n) = n + \lg n - 5$
Note that this formula holds for all n, n a power of 2, greater than or equal to the first base case hit, i.e. 4. It does not hold for n=1 or n=2.

Prove our result using induction:
Claim: $W(n) = n + \lg n - 5$ for $n=2^k$, $k>=2$ (#)
Proof of Claim:
Base Case: For n = 4, we have our recursive formula (**) gives W(4) = 1, and our claim formula (#) gives 1. They match!

Induction Hypothesis (IH)
The claim is true for all m such that $4 \le m < n$, and $m = 2^k$,
i.e. $W(m) = m + \lg m - 5$ for $m=2^k$, $k>=2$.

Induction Step
Prove the claim is true for n, where $n \ge 8$, $n = 2^k$ (note that we have already shown it is true for $n = 2^2$, so now we show it for $n >= 2^3$).

$W(n) = W(n/2) + n/2 + 1$ (by the recursive formula (***))
$= (n/2 + \lg(n/2) - 5) + n/2 + 1$ (since n/2 is a power of 2 >=4 and <n, we can apply the IH to W(n/2))
$= n/2 + (\lg n - \lg 2) - 5 + n/2 + 1$ (property of logs)
$= n + \lg n - 5$, as required.

c) $\Theta(n)$.

4 a)

Usual algorithm (improved):
Number of multiplications = $n^3 = 16^3$
Number of additions = $n^3 - n^2 = 16^3 - 16^2$
Total number of operations = $2*16^3 - 16^2 = 7{,}936$

Strassen's Algorithm:
Number of multiplications = $7^{\lg n} = 7^{\lg 16} = 7^4$
Number of additions/subtractions = $6*7^{\lg n} - 6n^2 = 6(7^{\lg 16}) - 6(16^2)$
Total number of operations = $7(7^4) - 6(16^2) = 15{,}271$

So, when counting the operations listed above, the usual algorithm would be faster for n=16.

b) From class, the recurrence relation for Strassen's algorithm counting multiplications with a threshold of 4 will be:

$W(n) = 7W(n/2)$ for $n>4$, n a power of 2
$W(4) = 4^3 = 64$
$W(2) = 2^3 = 8$
$W(1) = 1$

(Note: For $n<= 4$ we will use the usual algorithm to multiply the matrices, which requires n^3 multiplications).

5.
Input size: n, the number of keys in the list. We are assuming $n = 2^k$ for some $k >=0$.
Operation being counted: Comparison of keys.
Let K(n) represent the number of comparisons for the reverse sorted ordered list.
(Note: For this analysis, I am assuming that the numbers are distinct.)

Since the list is in reverse sorted order, as every Merge step, we will be merging two lists where one list has values that are all smaller than the other. Given 2 sorted lists of length n/2 where one list has values that are all smaller than those in the second list, the Merge algorithm will only do n/2 comparisons, rather than n-1 comparisons as in the worst-case. Thus from the Mergesort algorithm we have

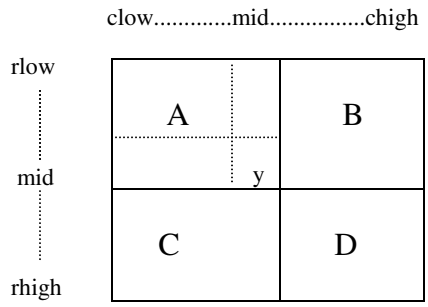
$K(n) = 0$ for $n=1$ (from the base case).
For $n>1$,
K(n) =(the work to sort the left half of the list) + (the work to sort the right half of the list) +
(the work to merge the two 1/2 lists)
$=W(n/2) + W(n/2) + n/2$
$= 2W(n/2) + n/2$.

6. a) There are many different solutions for this problem. Below I give two possible divide and conquer solutions. The first uses ideas very similar to those used in binary search. The second is a more efficient, slicker algorithm that runs in linear time.

Algorithm 1: A binary search type of algorithm

Idea of algorithm:
We have a row range rlow....rhigh, and a column range clow....chigh. To begin with this range will be 1...n for the rows, and 1...m for the columns. We look at the number in the table that is in the middle of both ranges, call it y. (See figure below—note that each of

the quadrants A, B, C and D would have size n/2 by n/2 for an n by n table, in the case where n is a power of 2).



If x=y, we are done. If x>y, then we know that x will not lie anywhere in quadrant A in the table (since the entries in the rows and columns are sorted). So we search each of quadrants B, D and C recursively for x. If x<y, then we know that x cannot lie in quandrant D, and we search quadrants A, B and C recursively. Finally, if the row range or the column range becomes empty (i.e. we have rlow>rhigh or clow>chigh) then we know x in not in the table.

Algorithm (in pseudocode):
Input: an n by m array *table*, sorted as described in the question, and a number x.
Output: a 2 element array *location*, where *location*[1] is the row location of x in table, and *location*[2] is the column location of x in *table*. If x is not in the table, these are 0.

```
void findx1(index rlow, index rhigh, index clow, index chigh, index location[])

{
  index rmid, cmid;

  if ((rlow>rhigh) || (clow>chigh)) {
        location[1]=0;
        location[2]=0;}
  else {
    rmid = ⌊(rlow + rhigh)/2⌋
    cmid = ⌊(clow + chigh)/2⌋
    if (x==table[rmid][cmid]){
          location[1] = rmid;
          location[2] = cmid; }
    else {
       if (x > table[rmid][cmid]){
          findx1(rmid+1, rhigh, clow, cmid, location);    \\look in quadrant C
          if (location[1] = = 0) \\not found in C
              findx1(rmid+1, rhigh, cmid+1, chigh, location);  \\look in quadrant D
```

```
            if (location[1] = = 0)  \\ not found in C or D
                findx1(rlow, rmid, cmid+1, chigh, location); }    \\look in quadrant B

        else }
            findx1(rmid+1, rhigh, clow, cmid, location);    \\look in quadrant C
             if (location[1] = = 0) \\ not found in C
                findx1(rlow, rmid, clow, cmid, location);  \\look in quadrant A
                if (location[1] = = 0) \\ not found in C or A
                    findx1(rlow, rmid, cmid+1, chigh, location); }    \\look in quadrant B

     }
}
```

Worst-case Analysis (we assume table is n by n, where n is a power of 2, i.e. $n = 2^k$, $k>=0$):

Input size: n, the number of rows and columns in table being searched
Basic operation counted: Comparison of x with keys
Input that gives worst case: x not in table, and bigger than all the numbers in table
Analysis:
Each of the quadrants has size n/2 by n/2, we search 3 quadrants recursively and do 2 comparisons outside of the recursive calls, so we have

$W(n) = 3W(n/2) + 2$, $n>=2$

$W(1) = 2$ (since when n is 1 and x is bigger than the single element in the table, then there are 2 comparisons, followed by 3 recursive calls where rlow>rhigh, each of which thus require 0 comparisons with x).

Back substitution:

$W(2^k) = 3W(2^{k-1}) + 2$
$= 3(3W(2^{k-2}) + 2) + 2$ (back substitute)
$=3^2W(2^{k-2}) +3*2 + 2$ (simplify)

.....

(ith line)
$=3^iW(2^{k-i}) + 2(3^{i-1} + 3^{i-2} + ... + 1)$

.....

(line k)
$=3^kW(2^{k-k}) + 2(3^{k-1} + 3^{k-2} + ... + 1)$
$=(3^k)*2 + 2(3^{k-1} + 3^{k-2} + ... + 1)$ (using W(1) = 2)
$=2(\Sigma (3^j: j=0 \text{ to } k))$ (simplify)
$= 2 ((3^{k+1} -1)/(3-1))$ (from Appendix A)
$= 3^{k+1} -1$ (simplify)
$=3* 3^k - 1$ (simplify)

$= 3*3^{\lg n} - 1$ (simplify)

$= 3*n^{\lg 3} - 1$ (using $a^{\lg b} = b^{\lg a}$)

$\approx 3*n^{1.585} - 1$

So this algorithm is $\approx\Theta(n^{1.585})$.

Algorithm 2: A linear-time algorithm.

Idea of algorithm: Suppose we have an array that is n by m.

We look at entry y = table[n][1]. If x=y, we are done. If x <y, then we can conclude that x is not in row n. So we now search the array consisting of the rows 1 to n-1, and columns 1 to m (i.e. the table with the nth row removed—so we have eliminated one row). If x > y, then we can conclude that x is not in the first column. So we now search the array consisting of the rows 1 to n, and the columns 2 to m (i.e. the table with the 1st column removed, so we have eliminated one column). So in general, given the search space in in the row range 1 to lastrow, and the column range firstcol to m, we look at the bottom left entry in this search space, i.e. table[lastrow][firstcol] and decide to either eliminate row rowlast or column firstcol from the space. Once either the number of rows or the number of columns of the table area we are searching is 0, we stop and conclude that x is not in the array.

Pseudocode for algorithm

Input: an n by m array *table*, sorted as described in the question, and a number x.

Output: a 2 element array *location*, where *location*[1] is the row location of x in table, and *location*[2] is the column location of x in *table*. If x is not in the table, these are 0.

```
void findx2(index lastrow, index firstcol, index location[])
{
if (lastrow==0) || (firstcol==m+1) {
    location[1]=0;
    location[2]=0; }
else
   if (x==table[lastrow][firstcol]){
       location[1]=lastrow;
       location[2]=lastcol;}
  else {
       if (x < table[lastrow][firstcol])
            lastrow=lastrow – 1;
        else
            firstcol=firstcol+1;
        findx2(lastrow, firstcol, location)  }
}
```

Worst-case analysis (Actually, for this worst case, I don't need to assume that n is a power of 2, so here I show the analysis for a general n by m table).

Input size: n+m, where n and m are the number of rows and columns in the table.
Basic operation counted: comparison of x with entries in table
Input that gives worst case: At each stage of the recursion, we either eliminate a row or a column from our search space, until either the number of rows left or the number of columns left becomes 0. Thus we get the most stages in the recursion if, at the very last stage, both the number of rows and the number of columns is 1 (so when we reach the base case, the number of rows and columns left are 1 and 0 or 0 and 1). An example of an input which creates this situation is one in which all of the entries in the table are 1, except for the nth row, where we have 2's in position 1 to m-1, and the mth column is all 4's, and the value of x is 3.

Analysis:
For our worst case input, we have
$W(n+m) = W(n+m-1) + 2$ for $n+m>=2$,
since at each stage of the recursion we decrease the total number of rows and columns by 1, and we do 2 comparisons outside of the recursive call.

For the base case, $W(1) = 0$ (i.e. when we have the number of columns is 1 and rows is 0).

Back substitution:

$W(n+m) = W(m+n-1) + 2$
$= W(m+n-2) + 2*2$ (back substitute)
$= W(m+n-3) + 2*3$ (back substitute)
⋮

(line i)
$= W(m+n-i) + 2*i$
⋮

(line m+n-1)
$=W(1) + 2*(m+n-1)$
$=2*(m+n-1)$. (since $W(1) = 0$)
So this algorithm is $\Theta(n+m)$.