# Assignment #2 - Solutions

1. (4.8 in text)

At room temp. (I'll take T= 25°C) we have,

$$E = k_B T = (1.381\times10^{-23}\,J/K)(298K)$$
$$= 4.12\times10^{-21}\,J\cdot\frac{1eV}{1.602\times10^{-19}J}$$
$$= 2.57\times10^{-2}\,eV$$

a) No, not enough available energy.

b) Yes, available energy is greater than the $10^{-4}$eV needed.

2. (5.4 in text)

To do this problem use the following results from Appendix C in the text,

$$\int_0^\infty x^{2n}e^{-\alpha x^2}dx = \frac{(2n)!}{n!\,2^{2n+1}}\sqrt{\frac{\pi}{\alpha^{2n+1}}} \qquad \text{(Eq. 2.1)}$$

$$\int_0^\infty x^{2n+1}e^{-\alpha x^2}dx = \frac{n!}{2\alpha^{n+1}} \qquad \text{(Eq. 2.2)}$$

We have,

$$f(v) = N v^2 \exp\left[-\frac{mv^2}{2k_BT}\right]$$

where $N$ is a normalization constant. We first let $\alpha = \frac{m}{2k_BT}$, and then solve for $N$ using,

$$1 = \int_0^\infty f(v)dv = N\int_0^\infty v^2 e^{-\alpha v^2}dv$$

Use Eq. 2.1 with $n=1$,

$$\int_0^\infty v^2 e^{-\alpha v^2}dv = \frac{2!}{1!\,2^3}\sqrt{\frac{\pi}{\alpha^3}} = \frac{1}{4}\sqrt{\frac{8k_B^3T^3\pi}{m^3}} = \frac{\sqrt{\pi}}{4}\left(\frac{2k_BT}{m}\right)^{3/2}$$

Thus,

$$N = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2k_BT}\right)^{3/2}$$

Now, for $\langle v \rangle$ we have,

$$\langle v \rangle = \int_0^\infty v f(v)\,dv = N\int_0^\infty v^3 e^{-\alpha v^2}\,dv, \quad \text{use Eq. 2.2 with } n=1,$$

$$= N\left[\frac{1!}{2\alpha^2}\right] = \frac{4}{\sqrt{\pi}}\cdot\frac{1}{2}\cdot\left(\frac{m}{2k_BT}\right)^{3/2}\left(\frac{m}{2k_BT}\right)^{-2}$$

$$= \frac{2}{\sqrt{\pi}}\sqrt{\frac{2k_BT}{m}} = \sqrt{\frac{8k_BT}{\pi m}}$$

So,

$$\boxed{\langle v \rangle = \sqrt{\frac{8k_BT}{\pi m}}}$$

For $\langle 1/v \rangle$ we have,

$$\langle 1/v \rangle = \int_0^\infty \frac{1}{v} f(v)\,dv = N\int_0^\infty v e^{-\alpha v^2}\,dv, \quad \text{use Eq. 2.2 with } n=0,$$

$$= N\left[\frac{0!}{2\alpha}\right] = \frac{4}{\sqrt{\pi}}\cdot\frac{1}{2}\cdot\left(\frac{m}{2k_BT}\right)^{3/2}\left(\frac{m}{2k_BT}\right)^{-1}$$

$$= \frac{2}{\sqrt{\pi}}\sqrt{\frac{m}{2k_BT}} = \sqrt{\frac{2m}{\pi k_BT}}$$

So,

$$\boxed{\langle 1/v \rangle = \sqrt{\frac{2m}{\pi k_BT}}}$$

Then,

$$\langle v \rangle\langle 1/v \rangle = \sqrt{\frac{8k_BT}{\pi m}}\cdot\sqrt{\frac{2m}{\pi k_BT}} = \frac{4}{\pi}$$

which is the desired result.

3. (5.7 in text)

We want the rms speed of Na at $T = 6000K$. The rms speed is,

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m}}$$

the mass of Na is,

$$m_{Na} = 23\ amu = 23\ g/mol \cdot \frac{1\ mol}{6.02\times10^{23}} \cdot \frac{1\ kg}{1000\ g}$$
$$= 3.82\times10^{-26}\ kg$$

So,

$$v_{rms} = \sqrt{\frac{3(1.381\times10^{-23}\ J/K)(6000K)}{(3.82\times10^{-26}\ kg)}} = 2551\ m/s = 2.55\ km/s$$

To get the Doppler broadening we use eq. 5.26 & 5.27 from the text,

$$\frac{\Delta\lambda^{FWHM}}{\lambda_0} = \frac{\Delta\nu^{FWHM}}{\nu_0} = 7.16\times10^{-7}\sqrt{\frac{T}{m}} \qquad Eq.\ 3.1$$

where T is in K & m is in <u>amu</u>. We want the frequency broadening, $\Delta\nu$, but we are given $\lambda_0$. We also have,

$$\nu_0 = c/\lambda_0$$

so, rearranging Eq. 3.1 for $\Delta\nu$ gives,

$$\Delta\nu = 7.16\times10^{-7}\sqrt{\frac{T}{m}} \cdot \frac{c}{\lambda_0}$$

Again, $T = 6000\ K$, but $m = 23\ amu$. We thus get,

$$\Delta\nu = 7.16\times10^{-7}\sqrt{\frac{6000K}{23amu}} \cdot \frac{3\times10^{8}\ m/s}{5900\times10^{-10}\ m} = 5.9\times10^{9}\ Hz$$
$$= 5.9\ GHz$$

4.

The fraction of gas molecules with speeds between $v$ and $v+dv$ is given by $f(v)dv$. Here $dv$ is an infinitesmal range, thus, we want,

$$\text{fraction of molecules between } v_1 \text{ \& } v_2 = F = \int_{v_1}^{v_2} f(v)dv$$

There is no analytic solution for the above integral, so it must be done numerically. What this means is that we split our interval into smaller intervals & make the approximation that $f(v)$ is constant over these intervals. As an example, say we have an interval $\Delta v = [v_0, v_n]$, to approximate $F$ we split $\Delta v$ into $n$ smaller intervals, $\Delta v_i$, i.e.,

$$F = \int_{v_I}^{v_F} f(v)dv = \int_{v_I}^{v_1} f(v)dv + \int_{v_1}^{v_2} f(v)dv + \ldots + \int_{v_{n-1}}^{v_F} f(v)dv$$

We now make the approximation that $f(v)$ is constant over each interval,

$$F = f(v_0)\int_{v_0}^{v_1} dv + f(v_1)\int_{v_1}^{v_2} dv + \ldots + f(v_{n-1})\int_{v_{n-1}}^{v_n} dv$$

$$= f(v_0)(v_1 - v_0) + f(v_1)(v_2 - v_1) + \ldots + f(v_{n-1})(v_n - v_{n-1})$$

If we have each small interval equal to $\frac{\Delta v}{n} = \frac{v_n - v_0}{n}$ we can write,

$$F = f(v_I)\frac{\Delta v}{n} + f(v_1)\frac{\Delta v}{n} + \ldots + f(v_{n-1})\frac{\Delta v}{n}$$

$$= \frac{\Delta v}{n}\left[f(v_0) + f(v_1) + \ldots + f(v_{n-1})\right]$$

$$= \frac{\Delta v}{n}\sum_{i=0}^{n-1} f(v_i)$$

We get back the exact solution by taking $n \to \infty$ such that $\frac{\Delta v}{n} \to dv$. Now, for the question at hand, we have our interval $\Delta v = [299, 301]$ m/s. We take $n=2$ for this example (could take $n=1$), so,

$$F = \int_{299}^{301} f(v)dv = \int_{299}^{300} f(v)dv + \int_{300}^{301} f(v)dv \approx f(299)\int_{299}^{300} dv + f(300)\int_{300}^{301} dv$$

$$= f(299)(300-299) + f(300)(301-300)$$

Plugging in the numbers gives us the fraction of $N_2$ molecules to be,

$$F = 0.003309$$
$$= 0.33\%$$

If we had chosen $n=1$ we would get,

$$F = f(299)(301-299) = 0.003304$$

and with $n=100$ we would get,

$$F = 0.003314$$

This shows that taking $n=1$ gives a good approximation.

5.

We want the temp. at which the rms speed of $N_2$ equals the Earth's escape velocity, i.e.,

$$v_{rms} = v_e$$

$$\sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{2GM}{r}}$$

Rearranging for T gives,

$$T = \frac{2GMm}{3k_Br} = \frac{2}{3}\frac{(6.67\times10^{-11} m^3 kg^{-1} s^{-2})(5.97219\times10^{24} kg)(4.65\times10^{-26} kg)}{(1.381\times10^{-23} J/K)(6.371\times10^{6} m)}$$

$$= 1.40\times10^{5} m/s$$