Assignment #1 - Solutions

1. (1.4 in text)

a) $^{2}C_{1} = \dfrac{2!}{(2-1)!\,1!} = 2$, There are 2 ways.

b) $^{20}C_{10} = \dfrac{20!}{10!\,(20-10)!} = 184756$, There are 184,756 ways.

c) $^{2\times10^{23}}C_{1\times10^{23}} = \dfrac{(2\times10^{23})!}{(1\times10^{23})!\,(2\times10^{23}-1\times10^{23})!} = \dfrac{(2\times10^{23})!}{[(1\times10^{23})!]^2}$

These are too big to calculate directly → Stirling's approximation:

$$\ln(2\times10^{23}!) \approx (2\times10^{23})\ln(2\times10^{23}) - 2\times10^{23} = 1.053\times10^{25}$$

$$\ln(1\times10^{23}!)^2 = 2\ln(1\times10^{23}!) \approx 2[1\times10^{23}\ln(1\times10^{23}) - 1\times10^{23}]$$
$$= 1.039\times10^{25}$$

So, $\ln\left({}^{2\times10^{23}}C_{1\times10^{23}}\right) = \ln(2\times10^{23}!) - \ln[(1\times10^{23}!)^2] \approx 1.386\times10^{23}$

and, $^{2\times10^{23}}C_{1\times10^{23}} \approx e^{1.386\times10^{23}} = 10^{6.02\times10^{22}}$

Thus, there are approximately $10^{6.02\times10^{22}}$ ways of arranging 10^{23} quanta among 2×10^{23} atoms.

2. (2.5 in text)

We assume $T_2 > T_1$ for simplicity (works either way). From conservation of energy we must have heat flow into body 1 equal to heat flow out of body 2, i.e.,

$$\Delta Q_1 + \Delta Q_2 = 0$$

$$C_1 \Delta T_1 + C_2 \Delta T_2 = 0$$

Both bodies have same final temps, so,

$$\Delta T_1 = T_F - T_1 \quad , \quad \Delta T_2 = T_F - T_2$$

We thus have,

$$C_1(T_F - T_1) + C_2(T_F - T_2) = 0$$

Rearrange for T_F,

$$\boxed{T_F = \frac{C_1T_1 + C_2T_2}{C_1 + C_2}}$$

We now assume $C_1 >> C_2$. Rewrite the above as,

$$T_F = \frac{C_1T_1 + C_2T_2}{C_1 + C_2} \cdot \frac{1/C_1}{1/C_1} = \frac{T_1 + C_2/C_1 \, T_2}{1 + C_2/C_1}$$

Since $C_2 << C_1$ we have $C_2/C_1 << 1$. We can thus expand the denominator above. With $x = C_2/C_1$,

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \approx 1 - x \quad (\text{since } x << 1)$$

We therefore have,

$$T_F = \frac{T_1 + C_2/C_1 \, T_2}{1 + C_2/C_1} \approx \left(T_1 + \frac{C_2}{C_1}T_2\right)\left(1 - \frac{C_2}{C_1}\right) = T_1 + \frac{C_2}{C_1}(T_2 - T_1)$$

Which is the desired result.

3. (3.3 in text)

We have the Poisson distribution,

$$P(x) = \frac{e^{-m} m^x}{x!}$$

We will use the power series representation of e^m,

$$e^m = \sum_{x=0}^{\infty} \frac{m^x}{x!}$$

a) We have,

$$\sum_{x=0}^{\infty} P(x) = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!} = e^{-m} \cdot e^{m} = 1$$

b) We have,

$$\sum_{x=0}^{\infty} xP(x) = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} \cdot x = e^{-m} \sum_{x=0}^{\infty} x \frac{m^x}{x!} = e^{-m} \sum_{x=1}^{\infty} x \frac{m^x}{x!}$$

$$= me^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

Let $k = x-1$,

$$\sum_{x=0}^{\infty} xP(x) = me^{-m} \sum_{k=0}^{\infty} \frac{m^k}{k!} = me^{-m} \cdot e^{m} = m$$

Thus, $$\langle x \rangle = \sum_{x=0}^{\infty} xP(x) = m$$

c) The average number of deaths/yr/corps (λ) is,

$$\lambda = (0)(109/200) + (1)(65/200) + (2)(22/200) + (3)(3/200) + (4)(1/200)$$

$$= 122/200$$

$$= 0.61$$

To calculate expected frequencies multiply probability of observing n deaths/yr/corps by number of observations. For $n=0$ we have,

$$\text{Expected Frequency} = 200 \cdot \frac{e^{-0.61} \cdot (0.61)^0}{0!} = 108.7$$

To calculate for $n \geq 5$ deaths/yr/corps use the fact that,

$$P(n \geq 5) = 1 - P(n < 5)$$

# of deaths per year per corps	Observed Frequency	Calculated Frequency
0	109	108.7
1	65	66.3
2	22	20.2
3	3	4.1
4	1	0.6
≥5	0	0.1
TOTAL	200	200

4. (3.6 in text)

This is almost identical to example 3.7 in text. We have,

$$Y = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$$

the mean of Y is thus,

$$\langle Y \rangle = \frac{1}{n}(\langle X_1 \rangle + \langle X_2 \rangle + \dots + \langle X_n \rangle)$$

All X_i have the same mean, so,

$$\langle Y \rangle = \frac{1}{n} \cdot n \langle X \rangle = \langle X \rangle$$

To find σ_y^2 we will use,

$$\sigma_y^2 = \langle Y^2 \rangle - \langle Y \rangle^2$$

Thus,

$$\langle Y^2 \rangle = \frac{1}{n^2} \langle (X_1 + X_2 + \dots + X_n)^2 \rangle$$

$$= \frac{1}{n^2} \langle X_1^2 + \dots + X_n^2 + X_1X_2 + X_2X_1 + X_1X_3 + \dots \rangle$$

$$= \frac{1}{n^2} \left[\langle X_1^2 \rangle + \dots + \langle X_n^2 \rangle + \langle X_1X_2 \rangle + \langle X_2X_1 \rangle + \langle X_1X_3 \rangle + \dots \right]$$

$$= \frac{1}{n^2} \left[n\langle X^2 \rangle + n(n-1)\langle X \rangle^2 \right]$$

So,

$$\sigma_y^2 = \frac{1}{n}\left[\langle x^2\rangle - (n-1)\langle x\rangle^2\right] - \langle x\rangle^2$$

$$= \frac{1}{n}\left[\langle x^2\rangle - (n-1)\langle x\rangle^2 - n\langle x\rangle^2\right]$$

$$= \frac{1}{n}\left[\langle x^2\rangle - \langle x\rangle^2\right]$$

$$= \frac{1}{n}\sigma_x^2$$

So,

$$\boxed{\sigma_y = \frac{1}{\sqrt{n}}\sigma_x}$$

5. (3.7 in text)

The binomial distribution is,

$$P(n,k) = \frac{n!}{k!(n-k)!}\,p^k(1-p)^{n-k}$$

a) Let $m = np$, so,

$$P(n,k) = \frac{n!}{(n-k)!\,k!}\left(\frac{m}{n}\right)^k\left(1-\frac{m}{n}\right)^{n-k}$$

$$= \left[\frac{n!}{(n-k)!\,n^k}\right]\cdot\frac{m^k}{k!}\left(1-\frac{m}{n}\right)^{n-k}$$

The term in square brackets has a limit as $n\to\infty$ of,

$$\lim_{n\to\infty}\frac{n!}{(n-k)!\,n^k} = \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}$$

$$= \lim_{n\to\infty}\frac{n^k + an^{k-1} + bn^{k-2} + \cdots + zn}{n^k} \quad \left(\begin{array}{c}a,b,\ldots,z\\ \text{just numbers}\end{array}\right)$$

$$= \lim_{n\to\infty} 1 + \cancel{\frac{a}{n}} + \cancel{\frac{b}{n^2}} + \cdots + \cancel{\frac{z}{n^{k-1}}} = 1$$

We also have the two terms,

$$\left(1-\frac{m}{n}\right)^{n-k} = \left(1-\frac{m}{n}\right)^{n}\left(1-\frac{m}{n}\right)^{-k}$$

Remember that,

$$e^{-\frac{m}{n}} = \sum_{x=0}^{\infty}\frac{(-m/n)^x}{x!} = 1 - \frac{m}{n} + \frac{m^2}{2n^2} - \dots$$

As $n \to \infty$, $\frac{m}{n} \to 0$ so we can take $e^{-m/n} \approx 1 - m/n$ and,

$$\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{n} = \left(e^{-m/n}\right)^{n} = e^{-m}$$

and,
$$\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{-k} = \lim_{n\to\infty}\left(e^{-m/n}\right)^{-k} = \lim_{n\to\infty} e^{mk/n} = e^{0} = 1$$

Thus,
$$\lim_{n\to\infty} P(n,k) = \frac{m^k}{k!}\left[\lim_{n\to\infty}\frac{n!}{(n-k)!\,n^k}\right]\left[\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{n-k}\right]$$

$$= \frac{m^k e^{-m}}{k!}$$

Which is what we wanted to show.

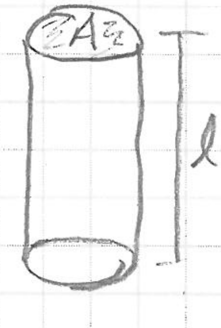
6.

We are given the length of the mercury column is $\ell = 5\,cm$ at the triple point of water, $T_T = 0.01°C$. We will assume that the ideal gas law (IGL) applies, so we can write,

$$PV = nRT$$

Here V is the volume of the Hg column, which we can write as,

$$V = \text{height} \cdot \text{area}$$
$$= \ell \cdot A$$



So, rearranging for ℓ in the IGL gives,

$$\ell = \frac{nRT}{PA}$$

For what we are interested in n, R, P, and A are all constant, so let,

$$\alpha = nR/PA$$

and we have,

$$\ell = \alpha T$$

We can determine α since $\ell = 5\,cm$ at $T = 0.01°C = 273.16K$,

$$\alpha = \frac{\ell}{T} = \frac{5cm}{273.16K} = 1.83 \times 10^{-2}\,cm/K$$

a) We want T when $\ell = 6.0\,cm$,

$$T = \frac{\ell}{\alpha} = \frac{6.0cm}{1.83 \times 10^{-2} cm/K} = 328\,K$$

b) We want ℓ when $T = 373.15K$,

$$\ell = \alpha T = (1.83 \times 10^{-2} \tfrac{cm}{K})(373.15K)$$
$$= 6.8\,cm$$

c) The length precision is $\Delta l = 0.01\text{cm}$. Since l is proportional to T we can simply write the temperature precision as,

$$\Delta T = \frac{\Delta l}{\alpha} = \frac{0.01\text{cm}}{1.83\times10^{-2}\text{cm/K}}$$

$$= 0.5\text{K}$$

This is much greater than the separation between the triple point & freezing point, thus the thermometer could not be used.