**Solutions to the final examination for MATH 1009* ABCD, Winter 2012.**

**PART A. MULTIPLE CHOICE QUESTIONS.**

**Answers: d a c d a d b a c b a c c b c d a**

1. The domain of the function $f(x) = \sqrt{(x+1)(x-4)}$ is

(a) $[-1, 4]$, or equivalently, $\{x : \ -1 \leq x \leq 4\}$.

(b) $(-1, 4)$, or equivalently, $\{x : \ -1 < x < 4\}$.

(c) $(-\infty - 1) \bigcup (4, \infty)$, or equivalently, $\{x : \ x < -1 \text{ and } x > 4\}$.

(d) $(-\infty - 1] \bigcup [4, \infty)$, or equivalently, $\{x : \ x \leq -1 \text{ and } x \geq 4\}$.

Answer: d)

2. . What is $\lim\limits_{x \to \infty} \dfrac{x^2 - x + 7}{x^3 + x - 5}$ ?

a) 0. b) 1. c) $\infty$. d) Does not exist.

Answer: a)

3. If $e^{x+2} = 7$, what is $x$ ?

a) $\ln 2 - 7$. b) $7 - \ln 2$. c) $\ln 7 - 2$. d) $2 - \ln 7$.

Answer : c)

4. Which of the following is equal to $\ln x - \ln x^3 + \ln x^2$ ?

a) $\ln(x - x^3 + x^2)$. b) $6 \ln x$. c) $2 \ln x$. d) 0.

Answer: d)

5. The expression $\dfrac{x^{2.4} \cdot (x^{-0.1})^4}{x^{-3} \cdot (x^2)^2}$ simplifies to

(a) $x$. (b) $x^{-0.5}$. (c) $x^3$. (d) 1.

Answer: a)

6. Which of the following is equal to $\log_{\frac{1}{2}} 16$ ?

a) $\frac{1}{4}$. b) $-\frac{1}{4}$. c) 4. d) $-4$.

Answer: d)

7. What is the slope of the curve $y = 2^x$ at $x = 2$?

a) $2\ln 2$. b) $4\ln 2$. c) 4. d) 0.

Answer: b)

8. Given that the cost function for a company is $C(x) = 60x - 0.025x^2 - 0.01x^3 + 2$ for $0 < x < 40$, find the **marginal cost** at a production level of $x = 20$ units.

a) 47. b) 49. c) 55.5. d) 1112.

Answer: a)

9. The derivative $f'$ of $f(x) = (3x^2+1)^{5/3}$ is

a) $10(3x^2+1)^{8/3}$ b) $\frac{5}{3}(3x^2+1)^{2/3}$ c) $10x(3x+1)^{2/3}$ d) $5x(3x^2+1)^{8/3}$

Answer: c)

10. The graph of the function $y = \frac{3x}{x-1}$ has

a) no horizontal asymptote and a vertical asymptote $x = 1$.

b) a horizontal asymptote $y = 3$ and a vertical asymptote $x = 1$.

c) a horizontal asymptote $y = 3$ and no vertical asymptote.

d) neither horizontal no vertical asymptote.

Answer: b)

11. What are the critical numbers of the function $f(x) = 3x^{1/3}$?

a) $x = 0$. b) $x = 3$. c) $x = \frac{1}{3}$. d) none.

Answer: a)

12. A company estimates that its daily cost function is $C(x) = x^3 - 6x^2 + 13x$ million dollars and its daily revenue function is $R(x) = 13x$ million dollars. What is the value of $x$ that maximizes the profit?

a) 1. b) 2. c) 4. d) 5.

Answer: c)

13. The second derivative $f''$ of $f(x) = \log_3(x-5)$, $(x > 5)$ is

a) $\dfrac{-1}{(x-5)^2}$ b) $\dfrac{1}{(x-5)}$ c) $\dfrac{-1}{\ln 3 \cdot (x-5)^2}$ d) $\dfrac{1}{\ln 3 \cdot (x-5)}$

Answer: c)

14. Let $f(x,y) = e^{2x+y}$. What is $f_{xx}(0,1)$?

a) $e$. b) $4e$. c) 0. d) none of the above.

Answer: b)

15. How long will it take an investment of \$5,000 to grow to \$6,000 if the investment earns interest at the rate of 4% compounded **continuously**?

a) 9.3years. b) 7.2 years. c) 4.5 years. d) 0.2 years.

Answer: c)

16. Consider the Cobb-Douglas production function $f(x,y) = 8x^{1/4}y^{3/4}$, where $x$ is the number of units of labour and $y$ is the number of units of capital. What is the marginal productivity of **capital** at $x = 16$, $y = 81$ ?

a) 432. b) $\frac{27}{4}$. c) 81. d) 4.

Answer: d)

17. The value of $\displaystyle\int_{-2}^{1} (2x+1)\,dx$ is

a) 0. b) 1. c) 4. d) 6.

Answer: a)

**PART B. For full marks, be sure to show all of your steps.**

**B1.** [ **16 Marks**] Consider the function $f(x) = x^3 - 3x + 2$.

[6] **(a)** Find the intervals where $f$ is increasing and those where $f$ is decreasing.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$. Since $f' < 0$ for all $x \in (-1, 1)$ , the function is decreasing for all such $x$. Since $f' > 0$ for all $x \in (-\infty, -1) \bigcup (1, \infty)$ , the function is increasing for all such $x$.

[3] **(b)** Find the relative extrema of $f$ and the values of $f$ at these points.

$f'(x) = 0 \;\Rightarrow\; x = -1, 1,\; f''(x) = 6x,$
$f''(-1) < 0 \;\Rightarrow\; f(-1) = 4$ is a relative maximum.
$f''(1) > 0 \;\Rightarrow\; f(1) = 0$ is a relative minimum.

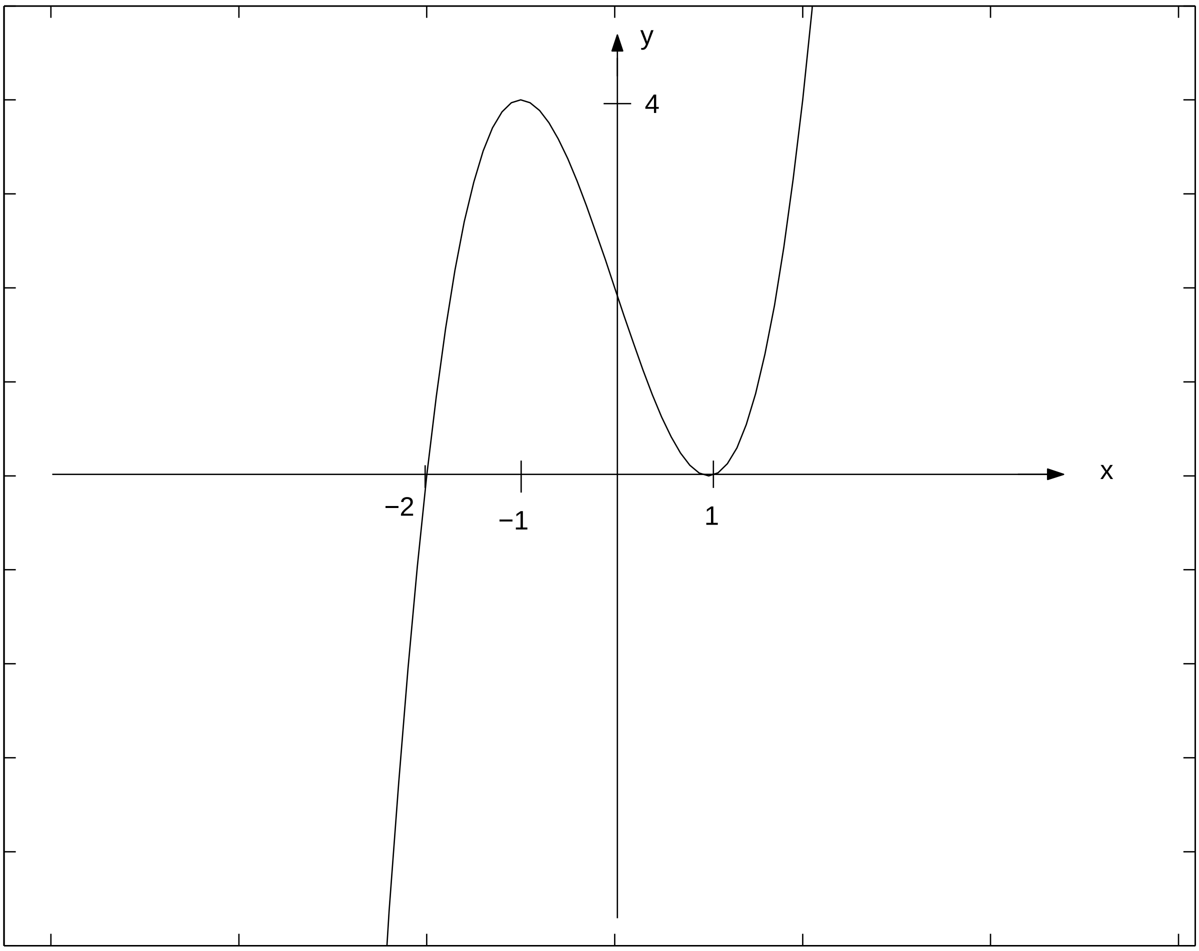
[3] **(c)** Determine the intervals where the graph of the function is concave up and those where it is concave down.

$f'' = 6x$. $f'' > 0$ when $x > 0$, so the graph is concave up (CU) for $x \in (0, \infty)$. $f'' < 0$ when $x < 0$, so the graph is concave down (CD) for $x \in (-\infty, 0)$.

[2] **(d)** Find the inflection points of $f$. (Give both the $x$- and the $y$-coordinates).

$f'' = 0$ when $x = 0$. Since $f'' = 0$ is changing its sign while moving across $x = 0$, therefore $x = 0$ is an inflection point. $f(0) = 2$.

[2] **(e)** Sketch the graph of $f$.




Graph of $f(x) = x^3 - 3x + 2$

**B2. [10 Marks]**

Use the method of Lagrange multipliers to minimize the function

$$f(x,y) = x^2 + 2y^2 - xy$$

subject to the constraint

$$2x + y = 22.$$

**Solution:**

$$F(x,y,\lambda) = x^2 + 2y^2 - xy + \lambda(2x + y - 22).$$

$$\begin{cases} F_x = 2x - \text{y} + 2\lambda = 0, & (1) \\ F_y = 4\text{y} - \text{x} + \lambda = 0, & (2) \\ F_\lambda = 2\text{x} + \text{y} - 22 = 0. & (3) \end{cases}$$

From (1) $\rightarrow$ $2\lambda = y - 2x$, so $\lambda = 0.5y - x$.
From (2) $\rightarrow$ $\lambda = x - 4y$.

Equate $\lambda$ to obtain $0.5y - x = x - 4y$, or $2x - 4.5y = 0$. This equation solved together with (3) yields

$$\begin{cases} 2\text{x} - 4.5\text{y} = 0 \\ 2\text{x} + \text{y} = 22 \end{cases} \Rightarrow 5.5y = 22, \ \ y = 4.$$

Then from (3), $x = \dfrac{22 - y}{2} = 9$. Thus, the constrained minimum is $f(9,4) = 77$.

**B3. [10 Marks]**

[4] **(a)** Find all the critical points of the function $f(x,y) = x^2 - 6xy + y^2 - 16y + 10$.
(Do NOT classify them.)

[6] **(b)** The function $f(x,y) = 2x^3 + y^2 - 6x + 4y$ has two critical points
$(1,-2)$ and $(-1,-2)$. Use the Second Derivative Test to classify each point as local maximum, local minimum or saddle.

**Solution:**

**(a)** $f_x = 2x - 6y = 0$, $f_y = -6x + 2y - 16 = 0$. Solving these two equations for $x$ and $y$ yields $y = -1$ and $x = -3$. Thus, the only critical point is $(-3,-1)$.

**(b)** The Second Derivative Test:

$f_{xx} = 12x$, $f_{yy} = 2$, $f_{xy} = 0$, so $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 12x \cdot 2 = 24x$.

$D(1,-2) = 24 > 0$, $f_{xx}(1,-2) = 12 > 0$, therefore $(1,-2)$ is a point of a local minimum.

$D(-1,-2) = -24 < 0$, therefore $(-1,-2)$ is a saddle point.

**B4. [13 Marks]**

**(a)** Evaluate the following integrals:

[2] (i) $\displaystyle\int \left(5x^4 + \frac{1}{x}\right) dx = 5\int x^4\, dx + \int \frac{1}{x}\, dx = 5 \cdot \frac{x^5}{5} + \ln|x| + C = x^5 + \ln|x| + C.$

[4] (ii) $\displaystyle\int_0^1 (3x-2)^4\, dx$

Let $u = 3x - 2$, then $\dfrac{du}{dx} = 3,\ dx = \dfrac{du}{3}.$

Now change the limits of integration: $x = 0 \Rightarrow u = 3 \cdot 0 - 2 = -2, \quad x = 1 \Rightarrow u = 3 \cdot 1 - 2 = 1.$
Thus,

$$\int_0^1 (3x-2)^4\, dx = \int_{-2}^1 u^4 \frac{du}{3} = \frac{1}{3}\int_{-2}^1 u^4\, du = \frac{1}{3} \cdot \frac{u^5}{5}\Big|_{-2}^1 = \frac{1}{15}[1^5 - (-2)^5] = \frac{33}{15}.$$

[3] (iii) $\displaystyle\int 6x^2\, e^{x^3+1}\, dx = \int 6x^2\, e^u \left(\frac{du}{3x^2}\right) = 2\int e^u\, du = 2e^u + C = 2e^{x^3+1} + C$

[4] **(b)** Consider the equation $xy + y^2 - x^2 = 10$, where $y = y(x)$ is defined implicitly as a function of $x$. Find $y' = \dfrac{dy}{dx}$.

Rewrite expression, stating that $y = y(x)$ is a function of $x$:

$x \cdot y(x) + (y(x))^2 - x^2 = 10.$

Differentiate both parts of the above expression with respect to $x$:

$1 \cdot y(x) + x \cdot y'(x) + 2y \cdot y'(x) - 2x = 0.$

Solve for $y'(x)$:
$x \cdot y'(x) + 2y(x) \cdot y'(x) = 2x - y(x), \qquad y'(x)(x + 2y) = 2x - y, \quad y'(x) = \dfrac{2x - y}{x + 2y}.$