University of Ottawa

Department of Mathematics and Statistics

MAT 1302D : Mathematical Methods II
Professor: Eric Hua

Third Midterm Exam – Version A

March 28, 2012

Surname ____________________ First Name ____________________

Student # ____________________

DGD # (1=VNR 5070; 2=LMX 124; 3=TBT 070; 4=DMS 1110)____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You have to show your work for each question (except for multiple choice questions).

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	Total
Maximum	4	4	6	8	8	30
Grade						

1. [**4 points**] For each of the following two questions, more than one answer may be correct. You should circle **all** of the correct answers. You will receive negative credit for each incorrect answer. But no negative mark for each question.

(a) [**2 points**] Which of the following sets are subspaces of $\mathbb{R}^3$?

$$U = \left\{ \begin{bmatrix} b-1 \\ b \\ c \end{bmatrix} \middle| b,\, c \in \mathbb{R} \right\}, \quad V = \left\{ \begin{bmatrix} bc \\ b \\ c \end{bmatrix} \middle| b,\, c \in \mathbb{R} \right\}, \quad W = \left\{ \begin{bmatrix} b+c \\ b-c \\ 0 \end{bmatrix} \middle| b,\, c \in \mathbb{R} \right\},$$

$$X = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}, \quad Y = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Solution: W, and X.

(b) [**2 points**] Let $A = \begin{bmatrix} h-1 & 2 \\ k & 1 \end{bmatrix}$. Which of the following statements imply that $rank(A) = 2$?

(a) $h = 4,\, k = 1$; (b) $h = 5,\, k = 2$; (c) $h = 7,\, k = 3$; (d) $h = -1,\, k = -1$;

(e) $h = -2,\, k = -2$; (f) $h = -5,\, k = -3$

Solution: (a), and (e). Since $rank(A) = 2$ iff $\det(A) = h - 1 - 2k \neq 0$.

2. [**4 points**] Let $z = 3 - 4i$, $w = 1 + 2i$.

(a) [**2 points**] Compute $\dfrac{z}{w}$. Your results should be in the form $a + ib$ where a and b are real numbers.

(b) [**2 points**] Compute $\overline{w} - |z|$. Your results should be in the form $a + ib$ where a and b are real numbers.

Solution:

(a) $\dfrac{z}{w} = \dfrac{3-4i}{1+2i} = \dfrac{(3-4i)(1-2i)}{(1+2i)(1-2i)} = \dfrac{-5-10i}{5} = -1 - 2i.$

(b) $\overline{w} - |z| = \overline{1+2i} - |3-4i| = 1 - 2i - 5 = -4 - 2i$

3. [**6 points**] Let $A = \begin{bmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 2 & 6 & 2 & -1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the row-reduced echelon form of A.

(a) [**1 points**] Find a basis for $ColA$, which is the column space of A.
(b) [**3 points**] Find a basis for $NulA$, which is the null space of A.
(c) [**2 points**] Find the dimensions of $ColA$ and $NulA$.

Solution: (a) By taking the pivot columns 1 and 4, a basis of $ColA$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(b) To find a basis for $NulA$, we solve $A\vec{x} = \vec{0}$. By the reduced row echelon form,

$$\begin{cases} x_1 + 3x_2 + x_3 + 3x_5 = 0, \\ x_4 + 4x_5 = 0. \end{cases} \Rightarrow \begin{cases} x_1 = -3x_2 - x_3 - 3x_5, \\ x_4 = -4x_5. \end{cases}$$

Thus

$$\vec{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

A basis for NulA will be

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

(c) dimColA=2, dimNulA=3.

4. [**8 points**]

(a) [**3 points**] Given $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -4 & 12 & -4 & 6 \\ 2 & -5 & 4 & -3 \\ -4 & 13 & 0 & 7 \end{bmatrix}$, find $\det A$ by row reduction.

Solution:

$$\det A = \begin{vmatrix} 1 & -3 & 2 & -4 \\ -4 & 12 & -4 & 6 \\ 2 & -5 & 4 & -3 \\ -4 & 13 & 0 & 7 \end{vmatrix} \xlongequal[R_4 \to R_4 + 4R_1]{\substack{R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 - 2R_1}} \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 4 & -10 \\ 0 & 1 & 0 & 5 \\ 0 & 1 & 8 & -9 \end{vmatrix}$$

$$\xlongequal{R_2 \leftrightarrow R_4} -\begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 8 & -9 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 4 & -10 \end{vmatrix} \xlongequal{R_3 \to R_3 - R_2} -\begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 8 & -9 \\ 0 & 0 & -8 & 14 \\ 0 & 0 & 4 & -10 \end{vmatrix}$$

$$\xlongequal{R_3 \to -\frac{1}{2}R_3} -(-2)\begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 8 & -9 \\ 0 & 0 & 4 & -7 \\ 0 & 0 & 4 & -10 \end{vmatrix} \xlongequal{R_4 \to R_4 - R_3} 2\begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 8 & -9 \\ 0 & 0 & 4 & -7 \\ 0 & 0 & 0 & -3 \end{vmatrix}$$

$$= 2(1)(1)(4)(-3) = -24.$$

(b) [**2 points**] Given the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 5 & -7 \\ 0 & 3 & 6 \end{bmatrix}$. What is the cofactor c_{13}?

Solution:

$$c_{13} = \det(A_{13}) = \begin{vmatrix} 2 & 5 \\ 0 & 3 \end{vmatrix} = 6.$$

(c) [**3 points**] Given 2×2 matrices A, B, C with $\det A = 2$, $\det B = 3$, $\det C = 5$. What is $\det(3AB^{-1}C^T)$?

Solution: $\det(3AB^{-1}C^T) = (3)^2(\det A)\left(\frac{1}{\det B}\right)(\det C) = 9(2)(1/3)(5) = 30.$

5. [**8 points**]

(a) [**2 points**] Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Determine whether $\vec{u}$ and $\vec{v}$ are eigenvectors of A.

Solution:

$$A\vec{u} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4\vec{u},$$

Thus $\vec{u}$ is an eigenvector of A.

$$A\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \neq \text{a multiple of } \vec{v},$$

Thus $\vec{v}$ is not an eigenvector of A.

(b) [**2 points**] Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find the characteristic polynomial.

Solution: $|A - \lambda I_2| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3.$

(c) [**4 points**] Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix}$. Find a basis for the eigenspace of A that corresponds to the eigenvalue $\lambda = 2$.

Solution: We solve the equation $(A - 2I_3)\vec{x} = \vec{0}$.

$$A - 2I_3 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_2 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \Rightarrow$$

$$x_1 + 2x_2 - x_3 = 0, \Rightarrow x_1 = -2x_2 + x_3.$$

Thus $(A - 2I_3)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus a basis of the eigenspace is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.